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DESIGN SENSITIVITY ANALYSIS OF NON-LINEAR STRUCTURES IN REGULAR AND CRITICAL STATES

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Abstract—A general design sensitivity formulation is introduced for non-linear elastic structures reaching critical equilibrium states. A discretized formulation is considered first and the sensitivity analysis is discussed for regular and critical states including analysis of post-critical behavior. Next, a variational approach is presented for beam and surface structures for which generalized stress and strain operator formulation is applied. The formulae for sensitivity of bifurcation loads are derived in terms of adjoint fields which are shown to be equivalent to post-buckling fields in the case of symmetric bifurcation.

1. INTRODUCTION

When a non-linear elastic structure is subjected to increasing external loads, it usually first passes through a regular deformation range associated with stable and unique response. Then the structure often reaches a critical state such as a limit or bifurcation point. Two related problems of interest are the immediate post-critical behavior and the variation of critical load with structural or imperfection parameters. For some cases, the character of critical point is not changed when small design variation occurs, but the critical load value is modified. In other cases, a design variation or geometric imperfection may induce disappearance of a critical point or change of its character. Such general questions of design sensitivity of critical equilibrium states will be discussed in the present paper. Our analysis will follow the previous work by Mróz (1987), Szefer *et al.* (1987), Mróz *et al.* (1985), Mróz and Haftka (1988), Cohen and Haftka (1989), Haftka *et al.* (1990), and Dems and Mróz (1989) on sensitivity of buckling loads and vibration frequencies of plates and shells with respect to variation of stiffness parameters and shape. A general variational approach to sensitivity analysis was presented by Haftka *et al.* (1990) and applied to surface structures within a generalized stress and strain formulation. The explicit sensitivity expressions were derived for variation of the critical load factor at the bifurcation point and of the vibration frequency. The close relation between design sensitivity and post-buckling analyses was indicated.

In Section 2, the design sensitivity of conservative discretized structures will be discussed in the regular case. In Section 3, the sensitivity of limit and bifurcation points will be considered. In Section 4 the variational approach will be applied to surface structures, thus paralleling our analysis of discretized systems. Section 5 is devoted to sensitivity of non-conservative discrete systems, and Section 6 discusses sensitivity of some simple structures.

2. SENSITIVITY ANALYSIS IN A REGULAR CASE

Consider an elastic discretized structure whose deformation is described by a set of generalized coordinates q_i and whose potential energy has the form

$$V = V(q_k, \lambda, s), \quad k = 1, \ldots, n, \tag{1}$$

where λ denotes the proportional loading parameter, and s is a design or imperfection parameter. The equilibrium equations are generated from (1)

$$\frac{\partial V}{\partial q_i} = V_i(q_k, \lambda, s) = 0, \quad i = 1, \ldots, n. \tag{2}$$

Consider the equilibrium path in the $n+1$ load-configuration space (Fig. 1). Let the progression parameter along this path be η, so that $\mathbf{q} = \mathbf{q}(\eta)$, $\lambda = \lambda(\eta)$. Then, at any equilibrium state $\mathbf{q}^0(\eta_0)$, $\lambda^0(\eta_0)$, we can write

$$q_i = q_i^0 + \dot{q}_i\Delta\eta + \tfrac{1}{2}\ddot{q}_i\Delta n^2 + \cdots,$$
$$\lambda = \lambda^0 + \dot{\lambda}\Delta\eta + \tfrac{1}{2}\ddot{\lambda}\Delta\eta^2 + \cdots, \tag{3}$$

where $\dot{q}_i, \ddot{q}_i, \ldots, \dot{\lambda}, \ddot{\lambda}, \ldots$ denote derivatives of q_i and λ with respect to η at $\mathbf{q} = \mathbf{q}^0$, $\lambda = \lambda^0$, $\eta = \eta_0$, and $\Delta\eta = \eta - \eta_0$.

Equations (3) specify the *load-deformation process* in the vicinity of a considered equilibrium state q_i^0, λ_0. Consider now the *structural transformation process* generated by the variation of s. This process can be conceived to occur separately or simultaneously with the deformation process. It can therefore be assumed

$$s = s^0 + \dot{s}\Delta\eta + \tfrac{1}{2}\ddot{s}\Delta\eta^2 + \cdots, \tag{4}$$

where $\dot{s}, \ddot{s}, \ldots,$ denote derivatives of s with respect to η at $\eta = \eta_0$. For load-deformation processes we set $s = s^0$, $\dot{s} = \ddot{s} = \cdots = 0$. On the other hand, for transformation processes we set $\lambda = \lambda^0$, $\dot{\lambda} = \ddot{\lambda} = \cdots = 0$. The progression parameter could be any generalized coordinate, for instance, $\eta = q_1$, or load factor, $\eta = \lambda$. For transformation processes one can set $\eta = s$, and then $\dot{s} = 1$, $\ddot{s} = \dddot{s} = \cdots = 0$.

Differentiating (2) one obtains a set of identity relations expressing equilibrium conditions associated with variation of configuration, loading and structural parameters, namely

$$V_{ij}\dot{q}_j + V_{i\lambda}\dot{\lambda} + V_{is}\dot{s} = 0, \tag{5}$$

$$V_{ij}\ddot{q}_j + V_{ijk}\dot{q}_j\dot{q}_k + 2V_{ij\lambda}\dot{\lambda}\dot{q}_j + 2V_{ijs}\dot{s}\dot{q}_j + 2V_{is\lambda}\dot{s}\dot{\lambda} + V_{i\lambda\lambda}\dot{\lambda}^2 + V_{iss}\dot{s}^2 + V_{i\lambda}\ddot{\lambda} + V_{is}\ddot{s} = 0, \tag{6}$$

and the third-order perturbation equation has the form

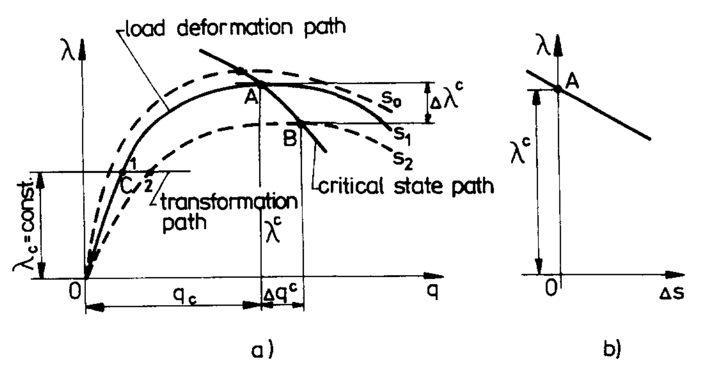

Fig. 1. (a) Load-deformation path of structure passing through limit point critical state path and transformation path. (b) Sensitivity diagram.

$$V_{ijkl}\dot{q}_j\dot{q}_k\dot{q}_l+3V_{ijk}\dot{q}_j\ddot{q}_k+V_{ij}\dddot{q}_j+3V_{ijk\lambda}\dot{q}_j\dot{q}_k\dot{\lambda}+3V_{ijks}\dot{q}_j\dot{q}_k\dot{s}+3V_{ij\lambda}\ddot{q}_j\dot{\lambda}+3V_{ijs}\ddot{q}_j\dot{s}$$
$$+3V_{ij\lambda}\dot{q}_j\ddot{\lambda}+3V_{ijs}\dot{q}_j\ddot{s}+V_{i\lambda}\dddot{\lambda}+V_{is}\dddot{s}+3V_{ij\lambda\lambda}\dot{q}_j\dot{\lambda}^2+3V_{ijss}\dot{q}_j\dot{s}^2+6V_{ij\lambda s}\dot{q}_j\dot{\lambda}\dot{s}$$
$$+3V_{i\lambda\lambda}\dot{\lambda}\ddot{\lambda}+3V_{iss}\dot{s}\ddot{s}+3V_{i\lambda s}\ddot{\lambda}\dot{s}+3V_{i\lambda s}\dot{\lambda}\ddot{s}+3V_{is\lambda\lambda}\dot{\lambda}^2\dot{s}+3V_{iss\lambda}\dot{\lambda}\dot{s}^2+V_{i\lambda\lambda\lambda}\dot{\lambda}^3+V_{isss}\dot{s}^3=0. \quad (7)$$

This set of perturbation equations allows for study of both deformation response, identification of critical points, and also of sensitivity to imperfection and design variations.

2.1. *Incremental load-deformation response*

Let us first discuss the regular case and assume that

$$V_{ij}\dot{q}_j \neq 0, \quad \text{or} \quad \det[V_{ij}] \neq 0. \quad (8)$$

With $\dot{s}=0$, eqn (5) provides the incremental equilibrium conditions along the load path.

$$V_{ij}q'_j+V_{i\lambda}\lambda'=0, \quad (9)$$

where $V_{ij}(q_k^0, \lambda^0, s^0)$ is the tangent stiffness matrix, and where to distinguish the derivatives with respect to loading parameters from other derivatives used in this paper, we denote them with primes instead of dots. When V_{ij} is positive definite, the incremental problem (9) is associated with the minimum principle of the incremental potential energy function

$$W(q'_i, \lambda') = \tfrac{1}{2}V_{ij}q'_iq'_j+V_{i\lambda}q'_i\lambda'. \quad (10)$$

In fact, the stationary condition for W leads to (9), that is

$$\delta W = (V_{ij}q'_j+V_{i\lambda}\lambda')\delta q'_i = 0, \quad (11)$$

where $\delta q'_i$ denotes a kinematically admissible variation. The absolute minimum of W occurs in the class of kinematically admissible q_i^k, thus in view of (9)

$$W(q_i'^k, \lambda')-W(q'_i, \lambda') = \tfrac{1}{2}V_{ij}q_i'^kq_j'^k-\tfrac{1}{2}V_{ij}q'_iq'_j+V_{i\lambda}(q_i'^k-q'_i)\lambda' = \tfrac{1}{2}V_{ij}(q_i'^k-q'_i)(q_j'^k-q'_j) \geqslant 0. \quad (12)$$

The second order incremental energy associated with (6) can be constructed in a similar way with the minimum condition specified by (12).

2.2. *Incremental transformation response: sensitivity analysis*

2.2.1. *Direct approach to sensitivity analysis.* Consider now the transformation path occurring at fixed λ but with varying design or imperfection parameter s. We denote the derivative of $\mathbf{q}$ with respect to s by $\mathbf{q}_s$, and then from (5) it follows that

$$V_{ij}q_{sj}+V_{is}=0. \quad (13)$$

The solution of (13) provides the sensitivity $\mathbf{q}_s$. Consider now an analytical function

$$G = G(\mathbf{q}, s) = G(\mathbf{q}(s), s). \quad (14)$$

The variation of G along the transformation path can be presented as follows:

$$G = G_0+\dot{G}\Delta s+\tfrac{1}{2}\ddot{G}\Delta s^2+\tfrac{1}{6}\dddot{G}\Delta s^3+\cdots, \quad (15)$$

where

$$\dot{G} = G_i q_{si} + G_s, \quad \ddot{G} = G_{ij} q_{si} g_{sj} + 2G_{is} q_{si} + G_{ss} + G_i q_{ssi}, \ldots, \tag{16}$$

and q_{ssi}, the second derivative of the displacement with respect to s, can be determined from (6), specialized to the case of $\eta = s$

$$V_{ij} q_{ssj} + V_{ijk} q_{sj} q_{sk} + 2V_{ijs} q_{sj} + V_{iss} = 0. \tag{17}$$

2.2.2. *Adjoint approach to sensitivity analysis.* Consider now the adjoint method. We derive the variation of G due to variation of s subject to the equilibrium constraint (2). Assuming λ is constant, the augmented function is

$$\bar{G} = G(q_j, s) - \mu_i V_i(q_j, s), \tag{18}$$

where $\boldsymbol{\mu}$ is a Lagrange multiplier vector. The variation of G is expressed as follows:

$$\dot{\bar{G}} = G_j q_{sj} + G_s - \mu_i V_{ij} q_{sj} - \mu_i V_{is} = (-V_{ij}\mu_i + G_j) q_{sj} + G_s - \mu_i V_{is}. \tag{19}$$

In order to eliminate the (computationally expensive) term q_{sj} in (19) we require the adjoint structure to satisfy

$$V_{ij}\mu_i - G_j = 0, \tag{20}$$

which requires a solution with the tangential stiffness matrix V_{ij}. Now the first order sensitivity G can be expressed as

$$\dot{G} = \dot{\bar{G}} = G_s - \mu_i V_{is}. \tag{21}$$

Instead of direct determination of q_{sj} from (13) followed by calculation of $\dot{G}$ from (16), we may use the adjoint state μ_i from (20) and calculate $\dot{G}$ from (21). The adjoint method is efficient when we need the derivative of G with respect to many variables, since only one adjoint solution is needed. The expression for $\ddot{G}$ in (16) requires the calculation of the second derivative field q_{ss}. Using the adjoint method we can eliminate this term. We start by specializing (6) to the transformation path and multiply by μ_i to get

$$V_{ijk}\mu_i q_{sj} q_{sk} + 2V_{ijs}\mu_i q_{sj} + V_{iss}\mu_i + V_{ij}\mu_i q_{ssj} = 0. \tag{22}$$

Using (20), the last term in (22) is equal to $G_j q_{ssj}$, and then $\ddot{G}$ from (16) may be written as

$$\ddot{G} = (-V_{ijk}\mu_i q_{sk} - 2V_{ijs}\mu_i + G_{ij} q_{si} + 2G_{js}) q_{sj} + G_{ss} - V_{iss}\mu_i. \tag{23}$$

3. SENSITIVITY ANALYSIS FOR CRITICAL STATES

Consider now a critical state satisfying the condition

$$V^c_{ij} q_{1j} = 0 \quad \text{or} \quad \det[V^c_{ij}] = 0, \tag{24}$$

where q_{1j} is the eigenvector of V^c_{ij} associated with the zero eigenvalue (assumed to be a simple eigenvalue). Multiplying (9) by q_{1i} we obtain

$$q_{1i} V^c_{i\lambda} \lambda' = 0, \tag{25}$$

so that either $V^c_{i\lambda} q_{1i} = 0$, or $\lambda' = 0$, at the critical state. The first case corresponds to bifurcation and the second to limit point. The values of the potential energy and coordinates at the critical state are denoted by the superscript c and the critical load path derivative is λ'_c.

3.1. *Limit point sensitivity and post-critical response*

For a limit point we have $\lambda_c' = 0$ as well as (24). From (9) we see that $\mathbf{q}'$ is a scalar multiple of the eigenvector $\mathbf{q}_1$, and its magnitude can be determined only by choosing a path parameter. For example, if the path parameter is chosen to be the jth coordinate, q_j, then we have an additional equation, $q_j' = 1$, which together with (9) defines /b$\mathbf{q}'$. Setting $\dot{s} = 0$ in (6) and using primes to denote load path derivatives we get

$$V_{ij}^c q_j'' + V_{ijk}^c q_j' q_k' + 2V_{ij\lambda}^c \lambda' q_j' + V_{i\lambda\lambda}^c \lambda_c'^2 + V_{i\lambda}^c \lambda_c'' = 0. \tag{26}$$

Multiplying (26) by q_i' and evaluating at the limit point ($\lambda = \lambda_c$, $\lambda_c' = 0$) we obtain with the aid of (9)

$$\lambda_c'' = -\frac{V_{ijk}^c q_i' q_j' q_k'}{V_{i\lambda}^c q_i'}. \tag{27}$$

Design or imperfection sensitivity can be calculated from (5) and (6) by considering the *critical state path* following limit points. Along that path we can have $s = \eta$ so that $\dot{s} = 1$, $\ddot{s} = 0$, etc. However, now both λ and s change simultaneously. Multiplying (5) by q_{1i} we can solve for $\dot{\lambda}$ which is the sensitivity of λ_c with respect to s

$$\lambda_{cs} = \dot{\lambda} = -\frac{V_{is}^c q_{1i}}{V_{i\lambda}^c q_{1i}}. \tag{28}$$

Similarly, by multiplying (6) by q_{1i} we obtain

$$\lambda_{css} = \ddot{\lambda} = -\frac{(V_{ijk}^c \dot{q}_j^c \dot{q}_k^c + 2V_{ij\lambda}^c \dot{q}_j^c \dot{\lambda}_c + 2V_{ijs}^c \dot{q}_j^c + 2V_{i\lambda s}^c \dot{\lambda}_c + V_{i\lambda\lambda}^c \dot{\lambda}_c^2 + V_{iss}^c) q_{1i}}{V_{i\lambda}^c q_{1i}}. \tag{29}$$

To evaluate $\dot{\mathbf{q}}^c$ appearing in (29) we need to solve eqn (5) which is singular at $\lambda = \lambda_c$. Equation (28) provides the consistency condition guaranteeing that (5) has a solution at $\lambda = \lambda_c$. However, we need one additional equation to make the solution of (5) unique. This extra condition is provided by differentiating (24) along the critical path to obtain

$$V_{ijk}^c q_{1j} \dot{q}_k^c + V_{ij\lambda}^c q_{1j} \dot{\lambda}_c + V_{ijs}^c q_{1j} + V_{ij}^c \dot{q}_{1j} = 0. \tag{30}$$

Multiplying by q_{1i} we have

$$V_{ijk}^c q_{1i} q_{1j} \dot{q}_k^c + V_{ij\lambda}^c q_{1i} q_{1j} \dot{\lambda}_c + V_{ijs}^c q_{1i} q_{1j} = 0 \tag{31}$$

which together with (5) provides the solution for $\dot{\mathbf{q}}^c$. Equation (30) can also be used to find $\dot{q}_{1j}$, the derivative of the limit-load eigenvector. However, this equation is also singular and must be supplemented by another relation derived from the normalization condition for the eigenvector. If that condition is

$$T_{ij} q_{1i} q_{1j} = 1, \tag{32}$$

where T_{ij} is a positive definite matrix, then by differentiating (32) we obtain

$$T_{ij} q_{1j} \dot{q}_{1i} = 0, \tag{33}$$

which is an orthogonality condition on $\dot{\mathbf{q}}_1$.

3.2. Bifurcation point: post-critical behaviour

Consider now the bifurcation point for which the following conditions are satisfied

$$V^c_{ij} q_{1j} = 0, \quad V^c_{i\lambda} q_{1i} = 0. \tag{34}$$

We denote the generalized displacements along the fundamental loading path by $\mathbf{q}_0$, and those along the post-critical path by $\mathbf{q}$, so that after bifurcation

$$q_j = q_{0j} + \Delta\eta q^p_{1j} + \tfrac{1}{2}\Delta\eta^2 q_{2j} + \tfrac{1}{6}\Delta\eta^3 q_{3j} + \cdots, \tag{35}$$

where η is the post-critical path parameter. We assume that q_{0j} is evaluated at the same load as q_j which brings in an indirect dependence of q_{0j} on η. This is

$$\begin{aligned} \dot{q}_j &= q'_{0j}\dot{\lambda} + q^p_{1j} + \Delta\eta q_{2j} + \tfrac{1}{2}\Delta\eta^2 q_{3j} + \cdots, \\ \ddot{q}_j &= q''_{0j}\dot{\lambda}^2 + q'_{0j}\ddot{\lambda} + q_{2j} + \Delta\eta q_{3j} + \cdots, \end{aligned} \tag{36}$$

where primes denote derivatives of the prebuckling state with respect to the load. The equations needed to obtain q'_{0j} and q''_{0j} are obtained from (5) and (6) by setting the path parameter to be λ

$$\begin{aligned} &V^c_{ij} q'_{0j} + V^c_{i\lambda} = 0, \\ &V^c_{ij} q''_{0j} + V^c_{ijk} q'_{0j} q'_{0k} + 2V^c_{ij\lambda} q'_{0j} + V^c_{i\lambda\lambda} = 0. \end{aligned} \tag{37}$$

Since V^c_{ij} is singular, q'_{0j} cannot be completely evaluated from eqn (37a), and an additional condition is required. This is obtained by multiplying eqn (37b) by q_{1i}

$$V^c_{ijk} q_{1i} q'_{0j} q'_{0k} + 2V^c_{ij\lambda} q_{1i} q'_{0j} + V^c_{i\lambda\lambda} q_{1i} = 0. \tag{38}$$

To find $\mathbf{q}^p_1$ we substitute $\dot{q}_j$ at $\Delta\eta = 0$ from (36) into (5) with $\dot{s} = 0$. Using (34) we get $V^c_{ij} q^p_{1j} = 0$, which indicates that $\mathbf{q}^p_1$ is an eigenvector of V^c_{ij}, so that it is a scalar multiple of $\mathbf{q}_1$. Since $\mathbf{q}_1$ is of indeterminate magnitude, we write

$$\mathbf{q}^p_1 = \mathbf{q}_1 = \alpha_c \bar{\mathbf{q}}_1 \tag{39}$$

where $\bar{\mathbf{q}}_1$ is the eigenvector $\mathbf{q}_1$ normalized to unit magnitude, and α_c will depend on the choice of η, as discussed later in this section. In order to derive the equation specifying q_{2j}, let us write (6) for the post-critical path, setting $\dot{s} = \ddot{s} = 0$ and using (36) with $\Delta\eta = 0$. We obtain

$$\begin{aligned} V^c_{ijk}(\dot{\lambda}_c q'_{0j} + q_{1j})(\dot{\lambda}_c q'_{0k} + q_{1k}) + 2V^c_{ij\lambda}(\dot{\lambda}_c q'_{0j} + q_{1j})\dot{\lambda}_c \\ + V^c_{i\lambda\lambda}\dot{\lambda}^2_c + V^c_{ij}(\dot{\lambda}^2_c q''_{0j} + \ddot{\lambda}_c q'_{0j} + q_{2j}) + V^c_{i\lambda}\ddot{\lambda}_c = 0. \end{aligned} \tag{40}$$

We now use (37) to obtain the equation specifying q_{2j}, namely

$$V^c_{ijk} q_{1j} q_{1k} + (2V^c_{ijk} q'_{0j} q_{1k} + 2V^c_{ij\lambda} q_{1j})\dot{\lambda}_c + V^c_{ij} q_{2j} = 0 \tag{41}$$

where we have made use of the symmetry $V^c_{ijk} = V^c_{ikj}$.

Multiplying eqn (41) by q_{1i} and solving for $\dot{\lambda}$, we get

$$\dot{\lambda}_c = -\frac{V^c_{ijk} q_{1i} q_{1j} q_{1k}}{2(V^c_{ijk} q_{1i} q'_{0j} q_{1k} + V^c_{ij\lambda} q_{1i} q_{1j})}. \tag{42}$$

Using (39), we can write (42) as

$$\dot{\lambda}_c = \frac{A}{2B}\alpha_c \tag{43}$$

where

$$A = V^c_{ijk}\bar{q}_{1i}\bar{q}_{1j}\bar{q}_{1k}$$
$$B = -(V^c_{ijk}\bar{q}_{1i}q_{0j}\bar{q}_{1k} + V^c_{ij\lambda}\bar{q}_{1i}\bar{q}_{1j}). \tag{44}$$

Assume now that η represents the post-critical path length traced by generalized coordinates. Using (36a) and (39), we have

$$\dot{\eta}_c = (\dot{q}_i\dot{q}_i)^{1/2} = [q'_{0i}q'_{0i}\dot{\lambda}_c^2 + \bar{q}_{1i}\bar{q}_{1i}\alpha_c^2 + 2\bar{q}_{1i}q'_{0i}\dot{\lambda}_c\alpha_c]^{1/2}$$
$$= \alpha_c\left[q'_{0i}q'_{0i}\left(\frac{A}{2B}\right)^2 + 1 + \frac{A}{B}\bar{q}_{1i}q'_{0i}\right]^{1/2} = 1 \tag{45}$$

where (43) was substituted into (45). Now, (45) provides

$$\alpha_c = \left[q'_{0i}q'_{0i}\left(\frac{A}{2B}\right)^2 + 1 + \frac{A}{B}\bar{q}_{1i}q'_{0i}\right]^{-1/2}. \tag{46}$$

Alternatively, one may select the coordinate q_1 as a path parameter. Then, we have

$$\dot{\eta} = \dot{q}_1 = q'_{01}\dot{\lambda}_c + \alpha_c\bar{q}_{11} = \alpha_c\left(\frac{A}{2B}q'_{01} + \bar{q}_{11}\right) = 1, \tag{47}$$

and

$$\alpha_c = \left[\bar{q}_{11} + \frac{A}{2B}q'_{01}\right]^{-1}. \tag{48}$$

When η represents the total path length in the $n+1$ dimensional load-configuration space, then

$$\dot{\eta} = (\dot{\lambda}_c^2 + \dot{q}_i\dot{q}_i)^{1/2} = 1 \tag{49}$$

and

$$\alpha_c = \left[(1 + q'_{0i}q'_{0i})\left(\frac{A}{2B}\right)^2 + 1 + \frac{A}{B}\bar{q}_{1i}q'_{0i}\right]^{-1/2}. \tag{50}$$

A more detailed discussion of evolution through critical paths can be found, for instance, in Riks (1979), Kouhia and Mikkola (1989), and Flores and Godoy (1992).

Equation (50) is useful for asymmetric bifurcation. For symmetric bifurcation $V^c_{ijk}q_{1i}q_{1j}q_{1k}$ is zero and we need to use q_{2j} to evaluate $\ddot{\lambda}_c$ from (7) specialized to the post-critical path

$$V^c_{ijkl}\dot{q}_j\dot{q}_k\dot{q}_l + 3V^c_{ijk}\dot{q}_j\ddot{q}_k + V^c_{ij}\dddot{q}_j + 3V^c_{ijk\lambda}\dot{q}_j\dot{q}_k\dot{\lambda}_c + 3V^c_{ij\lambda}\ddot{q}_j\dot{\lambda}_c + 3V^c_{ij\lambda}\dot{q}_j\ddot{\lambda}_c$$
$$+ V^c_{i\lambda}\dddot{\lambda}_c + 3V^c_{ij\lambda\lambda}\dot{q}_j\dot{\lambda}_c^2 + 3V^c_{i\lambda\lambda}\dot{\lambda}_c\ddot{\lambda}_c + V^c_{i\lambda\lambda\lambda}\dot{\lambda}_c^3 = 0. \tag{51}$$

The corresponding equation for the prebuckling path with λ as the path parameter is

$$V^c_{ijkl}q'_{0j}q'_{0k}q'_{0l} + 3V^c_{ijk}q'_{0j}q''_{0k} + V^c_{ij}q'''_{0j} + 3V^c_{ijk\lambda}q'_{0j}q'_{0k} + 3V^c_{ij\lambda}q''_{0j} + 3V^c_{ij\lambda\lambda}q'_{0j} + V^c_{i\lambda\lambda\lambda} = 0. \tag{52}$$

We start the process of evaluating $\ddot{\lambda}_c$ by the evaluation of q''_{0j} from (38). Since V^c_{ij} is singular we need an additional condition, obtained by evaluating (52) at the critical point and multiplying by q_{1i}

$$V^c_{ijkl}q_{1i}q'_{0j}q'_{0k}q'_{0l}+3V^c_{ijk}q_{1i}q'_{0j}q''_{0k}+3V^c_{ijk\lambda}q_{1i}q'_{0j}q'_{0k}+3V^c_{ij\lambda}q_{1i}q''_{0j}+3V^c_{ij\lambda\lambda}q_{1i}q'_{0j}+V^c_{i\lambda\lambda\lambda}q_{1i}=0. \tag{53}$$

Next we substitute from (36) into (51) multiplied by q_{1i} and evaluate it at the bifurcation point

$$\begin{aligned}V^c_{ijkl}q_{1i}(\dot\lambda_c q'_{0j}+q_{1j})(\dot\lambda_c q'_{0k}+q_{1k})(\dot\lambda_c q'_{0l}+q_{1l})+3V^c_{ijk}q_{1i}(\dot\lambda_c q'_{0j}+q_{1j})(\dot\lambda_c^2 q''_{0k}+\ddot\lambda_c q'_{0k}+q_{2k})\\ +3V^c_{ijk\lambda}\dot\lambda_c q_{1i}(\dot\lambda_c q'_{0j}+q_{1j})(\dot\lambda_c q'_{0k}+q_{1k})+3V^c_{ij\lambda}q_{1i}(\dot\lambda_c^2 q''_{0j}+\ddot\lambda_c q'_{0j}+q_{2j})\dot\lambda_c\\ +3V^c_{ij\lambda}q_{1i}(\dot\lambda_c q'_{0j}+q_{1j})\ddot\lambda_c+3V^c_{ij\lambda\lambda}q_{1i}(\dot\lambda_c q'_{0j}+q_{1j})\dot\lambda_c^2+3V^c_{i\lambda\lambda}q_{1i}\dot\lambda_c\ddot\lambda_c+V^c_{i\lambda\lambda\lambda}q_{1i}\dot\lambda_c^3=0.\end{aligned} \tag{54}$$

We subtract $\dot\lambda_c^3$ times (53) and $3\dot\lambda_c\ddot\lambda_c q_{1i}$ times (37b) to obtain

$$\begin{aligned}&V^c_{ijkl}(q_{1i}q_{1j}q_{1k}q_{1l}+3\dot\lambda_c q_{1i}q'_{0j}q_{1k}q_{1l}+3\dot\lambda_c^2 q_{1i}q'_{0j}q'_{0k}q_{1l})\\ &+3V^c_{ijk}(\dot\lambda_c q_{1i}q'_{0j}q_{2k}+\dot\lambda_c^2 q_{1i}q_{1j}q''_{0k}+\ddot\lambda_c q_{1i}q_{1j}q'_{0k}+q_{1i}q_{1j}q_{2k})+3V^c_{ijk\lambda}\dot\lambda_c(2\dot\lambda_c q_{1i}q'_{0j}q_{1k}+q_{1i}q_{1j}q_{1k})\\ &+3V^c_{ij\lambda}q_{1i}q_{2j}\dot\lambda_c+3V^c_{ij\lambda}q_{1i}q_{1j}\ddot\lambda_c+3V^c_{ij\lambda\lambda}q_{1i}q_{1j}\dot\lambda_c^2=0,\end{aligned} \tag{55}$$

so that

$$\ddot\lambda_c=-\frac{A+B\dot\lambda_c+C\dot\lambda_c^2}{3(V^c_{ij\lambda}q_{1i}q_{1j}+V^c_{ijk}q_{1i}q_{1j}q'_{0k})}, \tag{56}$$

where

$$\begin{aligned}A&=V^c_{ijkl}q_{1i}q_{1j}q_{1k}q_{1l}+3V^c_{ijk}q_{1i}q_{1j}q_{2k},\\ B&=3V^c_{ijkl}q_{1i}q'_{0j}q_{1k}q_{1l}+3V^c_{ijk}q_{1i}q'_{0j}q_{2k}+3V^c_{ijk\lambda}q_{1i}q_{1j}q_{1k}+3V^c_{ij\lambda}q_{1i}q_{2j},\\ C&=3V^c_{ijkl}q_{1i}q'_{0j}q'_{0k}q_{1l}+3V^c_{ijk}q_{1i}q_{1j}q''_{0k}+6V^c_{ijk\lambda}q_{1i}q'_{0j}q_{1k}.\end{aligned} \tag{57}$$

For the symmetric bifurcation point we have $\dot\lambda_c=0$ and (56) becomes

$$\ddot\lambda_c=-\frac{A}{3(V^c_{ij\lambda}q_{1i}q_{1j}+V^c_{ijk}q_{1i}q_{1j}q'_{0k})}. \tag{58}$$

The formulae (56) and (58) coincide in particular with those derived by Thompson and Hunt (1973) who considered the post-critical response using a local coordinate system sliding along the fundamental equilibrium path. The present derivation is presented in the global coordinate system and could be reduced to a local system by setting $q_{0i}=q'_{0i}=q''_{0i}=0$.

3.3. *Bifurcation point: sensitivity analysis*

Assume that the design variation occurs such that the bifurcation point is preserved (rather than the more common situation where it becomes a limit point) and the critical state path AB from the bifurcation point connects the consecutive bifurcation point satisfying (34)

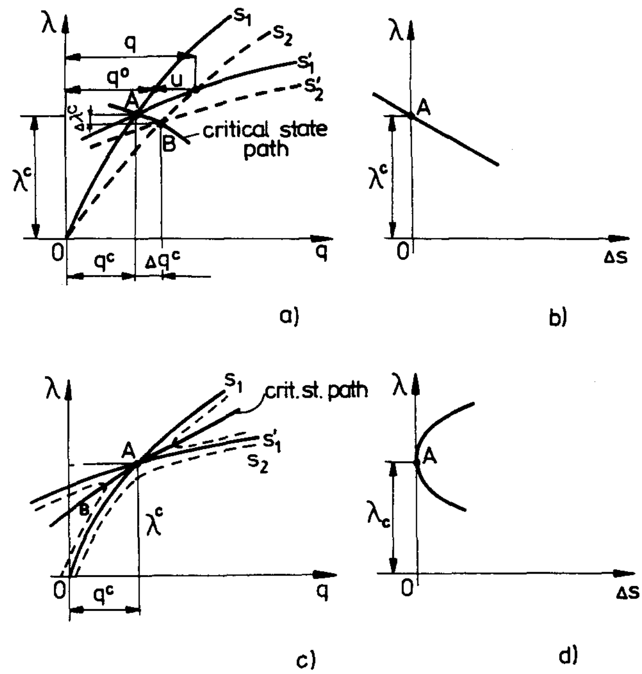

Fig. 2. (a) Load deformation path of structure passing through bifurcation point and critical state path for regular case. (b) Sensitivity diagram for regular case. (c) Load deformation path of structure and critical state path for singular case. (d) Sensitivity diagram for singular case.

(Fig. 2). Such a situation can occur, for instance, due to a plate or shell thickness variation or a laminate fiber angle modification in a composite structure that does not destroy the symmetry responsible for bifurcation.

For the critical state path we have from (5)

$$V^c_{ij}\dot{q}^c_j + V^c_{i\lambda}\lambda_{cs} + V^c_{is}\dot{s} = 0. \tag{59}$$

Since at the bifurcation point conditions (34) occur, then the *regular sensitivity case*, for which λ_{cs} is finite, requires that

$$q_{1i}V^c_{i\lambda} = 0, \quad q_{1i}V^c_{is} = 0. \tag{60}$$

Multiplying (59) by the eigenmode q_{1i}, one obtains formula (28). However, now both denominator and numerator vanish at the critical point, and to assess λ_{cs} one has to apply higher order equilibrium and critical state conditions. Consider evolution of the bifurcation point due to design variation. Along the critical path, we have

$$V^c_{ij}q_{1i}q_{1j} = 0, \quad V^c_i = 0, \tag{61}$$

where the first equation expresses the critical state condition and the second is the equilibrium condition. Considering the second equation (61) as constraint set on the critical state condition and introducing the Lagrange multipler μ_i, we can introduce an augmented critical condition

$$E = V^c_{ij}q_{1i}q_{1j} + \mu_i V^c_i = 0. \tag{62}$$

Differentiating (62) with respect to the critical path parameter, we obtain

$$V^c_{ijk}\dot{q}^c_k q_{1i} q_{1j} + \lambda_{cs} V^c_{ij\lambda} q_{1i} q_{1j} + V^c_{ijs} q_{1i} q_{1j} + \mu_i (V^c_{ij} \dot{q}^c_j + \lambda_{cs} V^c_{i\lambda} + V^c_{is}) = 0. \tag{63}$$

Note that along the critical path both the load and the stiffness may vary simultaneously so that

$$\dot{q}^c_k = \lambda_{cs} q'_{0k} + q_{0sk}, \tag{64}$$

and (63) may now be written as follows

$$(V^c_{ijk} q_{1i} q_{1j} + \mu_i V^c_{ik}) q_{0sk} + \lambda_{cs}[(V^c_{ijk} q'_{0k} + V^c_{ij\lambda}) q_{1i} q_{1j} + \mu_i (V^c_{ij} q'_{0j} + V^c_{i\lambda})] + V^c_{ijs} q_{1i} q_{1j} + \mu_i V^c_{is} = 0. \tag{65}$$

We have now two options for calculating λ_{sc} from (65). Setting the term containing $\boldsymbol{\mu}$ as vanishing one obtains

$$\lambda_{cs} = -\frac{V^c_{ijk} q_{1i} q_{1j} q_{0sk} + V^c_{ijs} q_{1i} q_{1j}}{(V^c_{ijk} q'_{0k} + V^c_{ij\lambda}) q_{1i} q_{1j}}. \tag{66}$$

This form requires the calculation of the prebuckling sensitivity $\mathbf{q}_{0s}$. In order to avoid this calculation, let us specify the adjoint field $\boldsymbol{\mu}$ by requiring that the coefficient of q_{0sk} in (65) vanishes, that is

$$V^c_{ik}\mu_i + V^c_{ijk} q_{1i} q_{1j} = 0. \tag{67}$$

By multiplying eqn (67) by q_{ik}, we note that $V^c_{ijk} q_{ii} q_{ij} q_{ik} = 0$, so that eqn (67) is consistent only for symmetric bifurcation.

In view of (37a) and (67), eqn (65) provides the alternative expression for λ_{cs}, namely

$$\lambda_{cs} = -\frac{\mu_i V^c_{is} + V^c_{ijs} q_{1i} q_{1j}}{(V^c_{ijk} q'_{0k} + V^c_{ij\lambda}) q_{1i} q_{1j}}. \tag{68}$$

Thus, the critical load sensitivity is expressed in terms of the eigenmode and the adjoint field μ_i satisying (67). Note that the field $\boldsymbol{\mu}$ is the same as the post buckling field $\mathbf{q}_2$ defined by (41) for the case of symmetric bifurcation. In fact, when $\lambda_c = 0$, eqn (41) specifying $\mathbf{q}_2$ is identical to (67).

Consider now the *singular sensitivity case*, typical for study of geometric imperfection sensitivity. In order to make distinction with the regular sensitivity case, denote the load variation by $\dot{\lambda}^m$, $\dot{\mathbf{q}}^c = \dot{\mathbf{q}}^m$, $\dot{s} = \dot{s}^m$ along the critical state path. We have now for cases of asymmetric bifurcation

$$V^c_{ij} q_{1j} = 0, \quad q_{1i} V^c_{i\lambda} = 0, \quad q_{1i} V^c_{is} \neq 0. \tag{69}$$

Then (5) evaluated at the critical state provides

$$V^c_{ij} \dot{q}^m_j + V^c_{i\lambda} \dot{\lambda}^m + V^c_{is} \dot{s}^m = 0. \tag{70}$$

Multiplying by q_{1i} we get

$$q_{1i}(V^c_{ij} \dot{q}^m_j + V^c_{i\lambda} \lambda_{cs} + V^c_{is} \dot{s}) = 0 \tag{71}$$

and in view of (69) we have

$$\dot{s}^m = 0. \tag{72}$$

Along the critical path issuing from the critical point, we have

$$\dot{q}_j^m = \dot{\lambda}^m q'_{0j} + q_{0sj}. \tag{73}$$

Substitute (73) into (70), which gives

$$(V_{ij}^c q'_{0j} + V_{i\lambda}^c)\dot{\lambda}^m + V_{ij}^c q_{0sj} = 0. \tag{74}$$

But the first term vanishes since it expresses the equilibrium condition along the prebuckling path, so we have

$$V_{ij}^c q_{0sj} = 0. \tag{75}$$

Since (75) is the critical state condition, identical to (69), we conclude that

$$q_{0sj} = q_{1j} = \alpha_c \bar{q}_{1j}, \tag{76}$$

where α_c is the scaling parameter with respect to the normalized eigenvector $\bar{q}_{1j}$. Thus the critical state path vector q_{0sj} coincides with the buckling mode q_{1j}.

Differentiating the critical state condition $V_{ij}^c q_{1j} = 0$, we have

$$V_{ijk}^c q_{1j} \dot{q}_k^m + V_{ij\lambda}^c q_{1j} \dot{\lambda}^m + V_{ijs}^c q_{1j} \dot{s}^m + V_{ij}^c \dot{q}_{1j} = 0. \tag{77}$$

Substitute

$$\dot{q}_j^m = \dot{\lambda}^m q'_{0j} + q_{0sj} = \dot{\lambda}^m q'_{0j} + q_{1j} \tag{78}$$

into (77) and multiply by q_{1i} to obtain

$$\dot{\lambda}^m = -\frac{V_{ijk}^c q_{1i} q_{1j} q_{1k}}{V_{ijk}^c q_{1i} q_{1j} q'_{0k} + V_{ij\lambda}^c q_{1i} q_{1j}}. \tag{79}$$

Comparing (79) with the expression (42) for post-critical derivative, we see that

$$\dot{\lambda}^m = 2\dot{\lambda}_c \tag{80}$$

and the scaling parameter α_c can be obtained from (46), (48) or (50) depending on the selection of path parameter. Consider now the second order equilibrium equation following from eqns (76) and (72)

$$V_{ij}^c \ddot{q}_j^m + V_{ijk}^c \dot{q}_j^m \dot{q}_k^m + 2V_{ij\lambda}^c \dot{q}_j^m \dot{\lambda}^m + V_{i\lambda\lambda}^c (\dot{\lambda}^m)^2 + V_{i\lambda}^c \ddot{\lambda}^m + V_{is}^c \ddot{s}^m = 0 \tag{81}$$

and the contracted form after multiplication by q_{1i}

$$V_{ijk}^c q_{1i} \dot{q}_j^m \dot{q}_k^m + 2V_{ij\lambda}^c q_{1i} \dot{q}_j^m \dot{\lambda}^m + V_{i\lambda\lambda}^c q_{1i} (\dot{\lambda}^m)^2 + V_{is}^c q_{1i} \ddot{s}^m = 0 \tag{82}$$

which provides:

$$\ddot{s}^m = -\frac{V_{ijk}^c q_{1i} \dot{q}_j^m \dot{q}_k^m + 2V_{ij\lambda}^c q_{1j} \dot{q}_j^m \dot{\lambda}^m + V_{i\lambda\lambda}^c q_{1i} (\dot{\lambda}^m)^2}{V_{is}^c q_{1i}}. \tag{83}$$

Assuming that $V_{i\lambda\lambda}^c = 0$, so that λ enters linearly into $V(q_i, \lambda, s)$, we have:

$$\ddot{s}^m = -\frac{V_{ijk}^c q_{1i} \dot{q}_j^m \dot{q}_k^m + 2V_{ij\lambda}^c q_{1j} \dot{q}_j^m \dot{\lambda}^m}{V_{is}^c q_{1i}}. \tag{84}$$

In view of (72) there is

$$\Delta s = s^m - s_c = \tfrac{1}{2}\ddot{s}^m \Delta\eta^2 + \tfrac{1}{6}\dddot{s}^m \Delta\eta^3 + \cdots$$
$$\Delta\lambda = \lambda^m - \lambda_c = \dot{\lambda}^m \Delta\eta + \tfrac{1}{2}\ddot{\lambda}^m \Delta\eta^2 + \cdots . \tag{85}$$

Neglecting higher order terms and eliminating $\Delta\eta$ in (85) we obtain

$$\Delta\lambda = \pm\dot{\lambda}_c^m \sqrt{\frac{2\Delta s}{\ddot{s}^m}} \tag{86}$$

which provides the square root singular sensitivity typical for geometric imperfection.

A similar discussion can be provided for the symmetric bifurcation point, cf. Thompson and Hunt (1977).

4. FUNCTIONAL FORM OF EQUATIONS

In this Section, we shall consider any surface or beam structure whose behavior is described in terms of generalized stress $\boldsymbol{\sigma}$, strain $\boldsymbol{\varepsilon}$, and displacement $\mathbf{u}$. Our derivation will provide an extension of the previous study by Haftka *et al.* (1990), and parallels the analysis of the previous section.

The equations governing large displacement and small strain response can also be written in a functional form introduced by Budiansky. The strain displacement relation is written as

$$\boldsymbol{\varepsilon} = \mathbf{L}_1(\mathbf{u}) + \tfrac{1}{2}\mathbf{L}_2(\mathbf{u}), \tag{87}$$

where $\mathbf{L}_1$ and $\mathbf{L}_2$ are first- and second-order homogeneous operators, respectively, in the displacement field $\mathbf{u}$. The variation of the strain is specified in terms of displacement variation as

$$\delta\boldsymbol{\varepsilon} = \mathbf{L}_1(\delta\mathbf{u}) + \mathbf{L}_{11}(\mathbf{u}, \delta\mathbf{u}), \tag{88}$$

where $\mathbf{L}_{11}$ is a symmetric bilinear operator, that is $\mathbf{L}_{11}(\mathbf{u}, \mathbf{v}) = \mathbf{L}_{11}(\mathbf{v}, \mathbf{u})$, defined by

$$\mathbf{L}_2(\mathbf{u}+\mathbf{v}) = \mathbf{L}_2(\mathbf{u}) + \mathbf{L}_2(\mathbf{v}) + 2\mathbf{L}_{11}(\mathbf{u}, \mathbf{v}). \tag{89}$$

In particular, (89) yields

$$\mathbf{L}_{11}(\mathbf{u}, \mathbf{u}) = \mathbf{L}_2(\mathbf{u}). \tag{90}$$

The linear stress–strain law is written as

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i), \tag{91}$$

where $\mathbf{D}$ is the stiffness tensor and $\boldsymbol{\varepsilon}^i$ is the initial strain tensor. We assume that the structure is loaded by a deformation independent load vector $\lambda\mathbf{f}$ where λ is a load amplitude parameter. The equations of equilibrium are written via the principle of virtual work as

$$\boldsymbol{\sigma} \bullet \delta\boldsymbol{\varepsilon} = \lambda\mathbf{f} \bullet \delta\mathbf{u}, \tag{92}$$

where $\bullet$ denotes a scalar product followed by integration over the structural domain

$$\boldsymbol{\sigma} \bullet \boldsymbol{\varepsilon} = \int \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} \, \mathrm{d}V. \tag{93}$$

We again consider the path parameter η, with eqn (4) for the response becoming

$$\mathbf{u} = \mathbf{u}^\circ + \dot{\mathbf{u}}\Delta\eta + \tfrac{1}{2}\ddot{\mathbf{u}}\Delta\eta^2 + \cdots,$$
$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^\circ + \dot{\boldsymbol{\varepsilon}}\Delta\eta + \tfrac{1}{2}\ddot{\boldsymbol{\varepsilon}}\Delta\eta^2 + \cdots,$$
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^\circ + \dot{\boldsymbol{\sigma}}\Delta\eta + \tfrac{1}{2}\ddot{\boldsymbol{\sigma}}\Delta\eta^2 + \cdots. \tag{94}$$

Differentiating (87), (88) and (89) with respect to the path parameter we get

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{L}_1(\dot{\mathbf{u}}) + \mathbf{L}_{11}(\mathbf{u}, \dot{\mathbf{u}}),$$
$$\dot{\boldsymbol{\sigma}} = \dot{\mathbf{D}}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i) + \mathbf{D}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^i),$$
$$\dot{\boldsymbol{\sigma}} \bullet \delta\boldsymbol{\varepsilon} + \boldsymbol{\sigma} \bullet \mathbf{L}_{11}(\dot{\mathbf{u}}, \delta\mathbf{u}) = \dot{\lambda}\mathbf{f} \bullet \delta\mathbf{u}. \tag{95}$$

Differentiating (95) once more we get

$$\ddot{\boldsymbol{\varepsilon}} = \mathbf{L}_1(\ddot{\mathbf{u}}) + \mathbf{L}_{11}(\mathbf{u}, \ddot{\mathbf{u}}) + \mathbf{L}_2(\dot{\mathbf{u}}),$$
$$\ddot{\boldsymbol{\sigma}} = \ddot{\mathbf{D}}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i) + 2\dot{\mathbf{D}}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^i) + \mathbf{D}(\ddot{\boldsymbol{\varepsilon}} - \ddot{\boldsymbol{\varepsilon}}^i),$$
$$\ddot{\boldsymbol{\sigma}} \bullet \delta\boldsymbol{\varepsilon} + 2\dot{\boldsymbol{\sigma}} \bullet \mathbf{L}_{11}(\dot{\mathbf{u}}, \delta\mathbf{u}) + \boldsymbol{\sigma} \bullet \mathbf{L}_{11}(\ddot{\mathbf{u}}, \delta\mathbf{u}) = \ddot{\lambda}\mathbf{f} \bullet \delta\mathbf{u}. \tag{96}$$

4.1. *Regular loading and stiffness sensitivity problems*

As in the discrete case we can specialize these equations for the case that η is a load parameter and for the case where η is a stiffness or imperfection parameter s. For the case of a load parameter variation, eqns (95) and (96) will provide the loading path response, namely

$$\boldsymbol{\varepsilon}' = \mathbf{L}_1(\mathbf{u}') + \mathbf{L}_{11}(\mathbf{u}, \mathbf{u}'),$$
$$\boldsymbol{\sigma}' = \mathbf{D}(\boldsymbol{\varepsilon}' - \boldsymbol{\varepsilon}^{i\prime}),$$
$$\boldsymbol{\sigma}' \bullet \delta\boldsymbol{\varepsilon} + \boldsymbol{\sigma} \bullet \mathbf{L}_{11}(\mathbf{u}', \delta\mathbf{u}) = \lambda'\mathbf{f} \bullet \delta\mathbf{u}, \tag{97}$$

and

$$\boldsymbol{\varepsilon}'' = \mathbf{L}_1(\mathbf{u}'') + \mathbf{L}_{11}(\mathbf{u}, \mathbf{u}'') + \mathbf{L}_2(\mathbf{u}'),$$
$$\boldsymbol{\sigma}'' = \mathbf{D}(\boldsymbol{\varepsilon}'' - \boldsymbol{\varepsilon}^{i\prime\prime}),$$
$$\boldsymbol{\sigma}'' \bullet \delta\boldsymbol{\varepsilon} + 2\boldsymbol{\sigma}' \bullet \mathbf{L}_{11}(\mathbf{u}', \delta\mathbf{u}) + \boldsymbol{\sigma} \bullet \mathbf{L}_{11}(\mathbf{u}'', \delta\mathbf{u}) = \lambda''\mathbf{f} \bullet \delta\mathbf{u}. \tag{98}$$

For the case of a variation of stiffness or imperfection parameter s we obtain the transformation path specified by the equations for the first order sensitivity

$$\boldsymbol{\varepsilon}_s = \mathbf{L}_1(\mathbf{u}_s) + \mathbf{L}_{11}(\mathbf{u}, \mathbf{u}_s),$$
$$\boldsymbol{\sigma}_s = \mathbf{D}_s(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i) + \mathbf{D}\boldsymbol{\varepsilon}_s,$$
$$\boldsymbol{\sigma}_s \bullet \delta\boldsymbol{\varepsilon} + \boldsymbol{\sigma} \bullet \mathbf{L}_{11}(\mathbf{u}_s, \delta\mathbf{u}) = 0, \tag{99}$$

and for the second order sensitivity

$$\boldsymbol{\varepsilon}_{ss} = \mathbf{L}_1(\mathbf{u}_{ss}) + \mathbf{L}_{11}(\mathbf{u}, \mathbf{u}_{ss}) + \mathbf{L}_2(\mathbf{u}_s),$$
$$\boldsymbol{\sigma}_{ss} = \mathbf{D}_{ss}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i) + 2\mathbf{D}_s\boldsymbol{\varepsilon}_s + \mathbf{D}\boldsymbol{\varepsilon}_{ss},$$
$$\boldsymbol{\sigma}_{ss} \bullet \delta\boldsymbol{\varepsilon} + 2\boldsymbol{\sigma}_s \bullet \mathbf{L}_{11}(\mathbf{u}_s, \delta\mathbf{u}) + \boldsymbol{\sigma} \bullet \mathbf{L}_{11}(\mathbf{u}_{ss}, \delta\mathbf{u}) = 0. \tag{100}$$

Let us note that the incremental loading problem (97) is associated with the minimum principle of the incremental potential energy

$$W^{(2)}(\mathbf{u}',\lambda') = \tfrac{1}{2}(\boldsymbol{\varepsilon}'-\boldsymbol{\varepsilon}^{i\prime})\bullet\mathbf{D}(\boldsymbol{\varepsilon}'-\boldsymbol{\varepsilon}^{i\prime})+\tfrac{1}{2}\boldsymbol{\sigma}\bullet\mathbf{L}_2(\mathbf{u}')-\lambda'\mathbf{f}\bullet\mathbf{u}'. \tag{101}$$

In fact, the stationary condition $\delta W = 0$ generates (97). In order to investigate the strong minimum condition, consider any kinematically admissible field $\mathbf{u}'_k$, $\boldsymbol{\varepsilon}'_k$. Then, we obtain in view of (97)

$$W^{(2)}(\mathbf{u}'_k,\lambda')-W^{(2)}(\mathbf{u}',\lambda') = \tfrac{1}{2}\Delta\boldsymbol{\varepsilon}'\bullet\mathbf{D}\Delta\boldsymbol{\varepsilon}'+\boldsymbol{\sigma}\bullet\tfrac{1}{2}\mathbf{L}_2(\Delta\mathbf{u}'), \tag{102}$$

where $\Delta\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}'_k-\boldsymbol{\varepsilon}'$, $\Delta\mathbf{u}' = \mathbf{u}'_k-\mathbf{u}'$. Assume that the incremental problem (97) satisfies the stability condition for any $\mathbf{u}'_1$, $\boldsymbol{\varepsilon}'$

$$(\boldsymbol{\varepsilon}'-\boldsymbol{\varepsilon}^{i\prime})\bullet\mathbf{D}(\boldsymbol{\varepsilon}'-\boldsymbol{\varepsilon}^{i\prime})+\boldsymbol{\sigma}\bullet\mathbf{L}_2(\mathbf{u}') = \lambda'\mathbf{f}\bullet\mathbf{u}' > 0. \tag{103}$$

Then, obviously the strong minimum of $W^{(2)}(\mathbf{u}',\lambda')$ occurs, and the right hand side of (102) is positive definite. The second order loading problem (98) is associated with the minimum principle of the second order potential energy, namely

$$W^{(4)}(\mathbf{u}'',\lambda'') = \tfrac{1}{2}(\boldsymbol{\varepsilon}''-\boldsymbol{\varepsilon}^{i\prime\prime})\bullet\mathbf{D}(\boldsymbol{\varepsilon}''-\boldsymbol{\varepsilon}^{i\prime\prime})+2\boldsymbol{\sigma}'\bullet\mathbf{L}_{11}(\mathbf{u}',\mathbf{u}'')+\tfrac{1}{2}\boldsymbol{\sigma}\bullet\mathbf{L}_2(\mathbf{u}'')-\lambda''\mathbf{f}\bullet\mathbf{u}''. \tag{104}$$

So the stationary condition $\delta W^{(4)} = 0$ is equivalent to the last equation (98). The strong minimum of $W^{(4)}(\mathbf{u}''_1\lambda'')$ occurs when (103) is satisfied for the second order states $\boldsymbol{\sigma}''$, $\boldsymbol{\varepsilon}''$ and $\mathbf{u}''$. The incremental sensitivity problem (99) corresponding to the transformation path is similarly associated with the minimum principle of the incremental potential energy

$$W^{(3)}(\mathbf{u}_s) = \tfrac{1}{2}\boldsymbol{\varepsilon}_s\bullet\mathbf{D}\boldsymbol{\varepsilon}_s+\tfrac{1}{2}\boldsymbol{\sigma}\bullet\mathbf{L}_2(\mathbf{u}_s)+\mathbf{D}_s\boldsymbol{\varepsilon}\bullet\boldsymbol{\varepsilon}_s, \tag{105}$$

where the last term plays the role of loading. The strong minimum of $W^{(3)}(\mathbf{u}_s)$ occurs provided

$$\boldsymbol{\varepsilon}_s\bullet\mathbf{D}\boldsymbol{\varepsilon}_s+\boldsymbol{\sigma}\bullet\mathbf{L}_2(\mathbf{u}_s) = -\mathbf{D}_s\boldsymbol{\varepsilon}\bullet\boldsymbol{\varepsilon}_s > 0 \tag{106}$$

for all kinematically admissible states $\mathbf{u}_s$ and $\boldsymbol{\varepsilon}_s$. Similarly as for the loading path, the second order potential energy corresponding to the transformation path is expressed as follows

$$W^{(3)}(\mathbf{u}_{ss}) = \tfrac{1}{2}\boldsymbol{\varepsilon}_{ss}\bullet\mathbf{D}\boldsymbol{\varepsilon}_{ss}+2\boldsymbol{\sigma}_s\bullet\mathbf{L}_{11}(\mathbf{u}_s,\mathbf{u}_{ss})+\tfrac{1}{2}\boldsymbol{\sigma}\bullet\mathbf{L}_2(\mathbf{u}_{ss})+2\mathbf{D}_s\boldsymbol{\varepsilon}\bullet\boldsymbol{\varepsilon}_{ss}+\mathbf{D}_{ss}\boldsymbol{\varepsilon}\bullet\boldsymbol{\varepsilon}_{ss}, \tag{107}$$

and its minimum occurs when (106) is satisfied for the states $\mathbf{u}_{ss}$ and $\boldsymbol{\varepsilon}_{ss}$.

So far we discussed the direct approach to sensitivity. In fact, after $\mathbf{u}_s$, $\boldsymbol{\varepsilon}_s$, $\boldsymbol{\sigma}_s$, and $\mathbf{u}_{ss}$, $\boldsymbol{\varepsilon}_{ss}$, $\boldsymbol{\sigma}_{ss}$ are calculated, then the first and second variations of a functional $G = G(\mathbf{u}, s)$ can be obtained as

$$G = G_0+\dot{G}\Delta s+\tfrac{1}{2}\ddot{G}\Delta s^2+\cdots, \tag{108}$$

where

$$\begin{aligned}\dot{G} &= \mathbf{G}_u\bullet\mathbf{u}_s+G_s,\\ \ddot{G} &= \mathbf{G}_{uu}\bullet\mathbf{u}_s\mathbf{u}_s+2\mathbf{G}_{us}\bullet\mathbf{u}_s+G_{ss}+\mathbf{G}_u\bullet\mathbf{u}_{ss}.\end{aligned} \tag{109}$$

In order to express the sensitivities in terms of adjoint variables, we impose constraints following from the equations of the prebuckling path, (87), (91) and (92), thus considering the augmented functional

$$G^* = G(\mathbf{u}(s),s)-\boldsymbol{\sigma}^a\bullet[\boldsymbol{\varepsilon}-\mathbf{L}_1(\mathbf{u})-\tfrac{1}{2}\mathbf{L}_2(\mathbf{u})]+\boldsymbol{\varepsilon}^a\bullet(\boldsymbol{\sigma}-\mathbf{D}\boldsymbol{\varepsilon}+\mathbf{D}\boldsymbol{\varepsilon}^i)$$
$$+\lambda\mathbf{f}\bullet\mathbf{u}-\boldsymbol{\sigma}\bullet[\mathbf{L}_1(\mathbf{u}^a)+\mathbf{L}_{11}(\mathbf{u},\mathbf{u}^a)]. \tag{110}$$

Differentiating G^* and rearranging the terms we obtain

$$\dot{G}^* = G_s + \mathbf{G}_u \bullet \mathbf{u}_s - \boldsymbol{\sigma}^a \bullet \boldsymbol{\varepsilon}_s - \boldsymbol{\sigma} \bullet \mathbf{L}_{11}(\mathbf{u}_s, \mathbf{u}^a) - \boldsymbol{\sigma}_s \bullet [\mathbf{L}_1(\mathbf{u}^a) + \mathbf{L}_{11}(\mathbf{u}, \mathbf{u}^a) - \boldsymbol{\varepsilon}^a] + \boldsymbol{\varepsilon}_s \bullet (\boldsymbol{\sigma}_a - \mathbf{D}\boldsymbol{\varepsilon}_a) - \boldsymbol{\varepsilon}^a \bullet \mathbf{D}_s(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i). \quad (111)$$

Equation (111) indicates that to avoid the calculation of prebuckling sensitivities we should define an adjoint problem satisfying

$$\begin{aligned} \boldsymbol{\varepsilon}^a &= \mathbf{L}_1(\mathbf{u}^a) + \mathbf{L}_{11}(\mathbf{u}, \mathbf{u}^a), \\ \boldsymbol{\sigma}^a &= \mathbf{D}\boldsymbol{\varepsilon}^a, \\ \boldsymbol{\sigma}^a \bullet \delta\boldsymbol{\varepsilon} + \boldsymbol{\sigma} \bullet \mathbf{L}_{11}(\mathbf{u}^a, \delta\mathbf{u}) &= \mathbf{G}_u \bullet \delta\mathbf{u}. \end{aligned} \quad (112)$$

Then the terms containing $\mathbf{u}_s$, $\boldsymbol{\sigma}_s$ and $\boldsymbol{\varepsilon}_s$ will disappear, and the sensitivity is expressed as follows

$$\dot{G}^* = \dot{G} = G_s - \mathbf{D}_s(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i) \bullet \boldsymbol{\varepsilon}^a. \quad (113)$$

The second order sensitivity is derived by using $\delta\mathbf{u} = \delta\mathbf{u}_{ss}$ in (112c) and $\delta\mathbf{u} = \delta\mathbf{u}^a$ in (100c). Then with some algebra (109) yields

$$\ddot{G} = \mathbf{G}_{uu} \bullet \mathbf{u}_s\mathbf{u}_s + 2\mathbf{G}_{us} \bullet \mathbf{u}_s + G_{ss} - 2\mathbf{D}_s\boldsymbol{\varepsilon}_s \bullet \boldsymbol{\varepsilon}^a - \mathbf{D}_{ss}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i) \bullet \boldsymbol{\varepsilon}^a - 2\boldsymbol{\sigma}_s \bullet \mathbf{L}_{11}(\mathbf{u}^a, \mathbf{u}_s) - \boldsymbol{\sigma}^a \bullet \mathbf{L}_2(\mathbf{u}_s). \quad (114)$$

4.2. *Critical state sensitivity*

At a critical point eqns (97) become singular and cannot be solved for the load-incremental state. The buckling load satisfies the homogeneous form of these equations

$$\begin{aligned} \boldsymbol{\varepsilon}_1 &= \mathbf{L}_1(\mathbf{u}_1) + \mathbf{L}_{11}(\mathbf{u}, \mathbf{u}_1), \\ \boldsymbol{\sigma}_1 &= \mathbf{D}\boldsymbol{\varepsilon}_1, \\ \boldsymbol{\sigma}_1 \bullet \delta\boldsymbol{\varepsilon} + \boldsymbol{\sigma} \bullet \mathbf{L}_{11}(\mathbf{u}_1, \delta\mathbf{u}) &= 0, \end{aligned} \quad (115)$$

where, as before, a subscript 1 denotes the buckling mode. Substituting $\delta\mathbf{u} = \mathbf{u}_1$ into (97c) we get

$$\boldsymbol{\sigma}' \bullet \boldsymbol{\varepsilon}_1 + \boldsymbol{\sigma} \bullet \mathbf{L}_{11}(\mathbf{u}', \mathbf{u}_1) = \lambda' \mathbf{f} \bullet \mathbf{u}_1. \quad (116)$$

Using (97b) and (115b) this becomes

$$\boldsymbol{\sigma}_1 \bullet (\boldsymbol{\varepsilon}' - \boldsymbol{\varepsilon}^{i\prime}) + \boldsymbol{\sigma} \bullet \mathbf{L}_{11}(\mathbf{u}', \mathbf{u}_1) = \lambda' \mathbf{f} \bullet \mathbf{u}_1. \quad (117)$$

Finally, substituting $\delta\mathbf{u} = \mathbf{u}'$ in (115c) and subtracting from (117) we get

$$\boldsymbol{\sigma}_1 \bullet \boldsymbol{\varepsilon}^{i\prime} + \lambda' \mathbf{f} \bullet \mathbf{u}_1 = 0. \quad (118)$$

When the initial strain is independent of the loading, $\boldsymbol{\varepsilon}^{i\prime} = 0$, and (118) is satisfied at a limit load by $\lambda' = 0$. At a bifurcation point $\lambda' \neq 0$ and so we have $\mathbf{f} \bullet \mathbf{u}_1 = 0$. In the following we limit ourselves to the case when $\boldsymbol{\varepsilon}^{i\prime} = 0$.

Limit point. At a limit point $\lambda' = 0$, and so by comparing (97) and (115) we note that $\mathbf{u}'$ is a scalar multiple of the buckling mode $\mathbf{u}_1$. Its magnitude can be determined by selecting the path parameter. For example, if the path parameter is selected to be one of the components of $\mathbf{u}$, then the corresponding components of $\mathbf{u}'$ is equal to 1. Setting $\delta\mathbf{u} = \mathbf{u}'$ in (98c) we get

$$\boldsymbol{\sigma}''\bullet\boldsymbol{\varepsilon}'+2\boldsymbol{\sigma}'\bullet\mathbf{L}_2(\mathbf{u}')+\boldsymbol{\sigma}\bullet\mathbf{L}_{11}(\mathbf{u}'',\mathbf{u}') = \lambda''\mathbf{f}\bullet\mathbf{u}'. \tag{119}$$

Similarly, setting $\delta\mathbf{u} = \mathbf{u}''$ in (97c) and using (98a) we get

$$\boldsymbol{\sigma}'\bullet[\boldsymbol{\varepsilon}''-\mathbf{L}_2(\mathbf{u}')]+\boldsymbol{\sigma}\bullet\mathbf{L}_{11}(\mathbf{u}'',\mathbf{u}') = 0. \tag{120}$$

From (97b) and (98b) we note that $\boldsymbol{\sigma}''\bullet\boldsymbol{\varepsilon}' = \boldsymbol{\sigma}'\bullet\boldsymbol{\varepsilon}''$, so that by subtracting (120) from (119) we get

$$\lambda'' = \frac{3\boldsymbol{\sigma}'\bullet\mathbf{L}_2(\mathbf{u}')}{\mathbf{f}\bullet\mathbf{u}'}, \tag{121}$$

which gives us the curvature at the limit point.

Design or imperfection sensitivity can be calculated from eqn (95) by considering a *critical state* path following limit points. Along the path we have $s = \eta$, $\dot{\lambda} = \lambda_{cs}$, and $\dot{\mathbf{D}} = \mathbf{D}_s$. Substituting $\delta\mathbf{u} = \mathbf{u}_1$ into (95c) and using (95b) we get

$$(\mathbf{D}\dot{\boldsymbol{\varepsilon}}+\mathbf{D}_s\boldsymbol{\varepsilon}^c-\mathbf{D}_s\boldsymbol{\varepsilon}^i)\bullet\boldsymbol{\varepsilon}_1+\boldsymbol{\sigma}^c\bullet\mathbf{L}_{11}(\dot{\mathbf{u}},\mathbf{u}_1) = \lambda_{cs}\mathbf{f}\bullet\mathbf{u}_1, \tag{122}$$

where the superscript c denotes evaluation at the limit point. Similarly, substituting $\delta\mathbf{u} = \dot{\mathbf{u}}$ into (115c) and using (115b) we get

$$\mathbf{D}\boldsymbol{\varepsilon}_1\bullet\dot{\boldsymbol{\varepsilon}}+\boldsymbol{\sigma}^c\bullet\mathbf{L}_{11}(\mathbf{u}_1,\dot{\mathbf{u}}) = 0. \tag{123}$$

Subtracting (123) from (122) we get

$$\lambda_{cs} = \frac{\mathbf{D}_s(\boldsymbol{\varepsilon}^c-\boldsymbol{\varepsilon}^i)\bullet\boldsymbol{\varepsilon}_1}{\mathbf{f}\bullet\mathbf{u}_1}. \tag{124}$$

We evaluate the second derivative in a similar manner, starting by setting $\delta\mathbf{u} = \mathbf{u}_1$ in (96c) and using (96b)

$$(\mathbf{D}_{ss}\boldsymbol{\varepsilon}^c-\mathbf{D}_{ss}\boldsymbol{\varepsilon}^i+2\mathbf{D}_s\dot{\boldsymbol{\varepsilon}}+\mathbf{D}\ddot{\boldsymbol{\varepsilon}})\bullet\boldsymbol{\varepsilon}_1+2\dot{\boldsymbol{\sigma}}\bullet\mathbf{L}_{11}(\dot{\mathbf{u}},\mathbf{u}_1)+\boldsymbol{\sigma}^c\bullet\mathbf{L}_{11}(\ddot{\mathbf{u}},\mathbf{u}_1) = \lambda_{css}\mathbf{f}\bullet\mathbf{u}_1. \tag{125}$$

Next we set $\delta\mathbf{u} = \ddot{\mathbf{u}}$ in (115c) and use (115b) to obtain

$$\mathbf{D}\boldsymbol{\varepsilon}_1\bullet[\ddot{\boldsymbol{\varepsilon}}-\mathbf{L}_2(\dot{\mathbf{u}})]+\boldsymbol{\sigma}^c\bullet\mathbf{L}_{11}(\mathbf{u}_1,\ddot{\mathbf{u}}) = 0. \tag{126}$$

Subtracting (126) from (125) we obtain

$$\lambda_{css} = \frac{(\mathbf{D}_{ss}\boldsymbol{\varepsilon}^c-\mathbf{D}_{ss}\boldsymbol{\varepsilon}^i+2\mathbf{D}_s\dot{\boldsymbol{\varepsilon}})\bullet\boldsymbol{\varepsilon}_1+\mathbf{D}\boldsymbol{\varepsilon}_1\bullet\mathbf{L}_2(\dot{\mathbf{u}})+2\dot{\boldsymbol{\sigma}}\bullet\mathbf{L}_{11}(\dot{\mathbf{u}},\mathbf{u}_1)}{\mathbf{f}\bullet\mathbf{u}_1}. \tag{127}$$

To evaluate (127) we need to solve (95) for $\dot{\mathbf{u}}$, $\dot{\boldsymbol{\varepsilon}}$, and $\dot{\boldsymbol{\sigma}}$ at $\lambda = \lambda_c$, where the system is singular. We need one additional condition to make the solution unique. This is obtained by first differentiating (115) along the critical path and setting $\delta\mathbf{u} = \mathbf{u}_1$

$$\begin{aligned}
\dot{\boldsymbol{\varepsilon}}_1 &= \mathbf{L}_1(\dot{\mathbf{u}}_1)+\mathbf{L}_{11}(\dot{\mathbf{u}},\mathbf{u}_1)+\mathbf{L}_{11}(\mathbf{u}^c,\dot{\mathbf{u}}_1),\\
\dot{\boldsymbol{\sigma}}_1 &= \mathbf{D}_s\boldsymbol{\varepsilon}_1+\mathbf{D}\dot{\boldsymbol{\varepsilon}}_1,\\
\dot{\boldsymbol{\sigma}}_1&\bullet\boldsymbol{\varepsilon}_1+\boldsymbol{\sigma}_1\bullet\mathbf{L}_{11}(\dot{\mathbf{u}},\mathbf{u}_1)+\dot{\boldsymbol{\sigma}}\bullet\mathbf{L}_2(\mathbf{u}_1)+\boldsymbol{\sigma}^c\bullet\mathbf{L}_{11}(\dot{\mathbf{u}}_1,\mathbf{u}_1) = 0.
\end{aligned} \tag{128}$$

Next we set $\delta\mathbf{u} = \dot{\mathbf{u}}_1$ in (115c) and use (115b) to have

$$\mathbf{D}\boldsymbol{\varepsilon}_1 \bullet [\dot{\boldsymbol{\varepsilon}}_1 - \mathbf{L}_{11}(\dot{\mathbf{u}}, \mathbf{u}_1)] + \boldsymbol{\sigma} \bullet \mathbf{L}_{11}(\mathbf{u}_1, \dot{\mathbf{u}}_1) = 0. \tag{129}$$

We now use (128b) and subtract (129) from (128c) to get

$$\mathbf{D}_s\boldsymbol{\varepsilon}_1 \bullet \boldsymbol{\varepsilon}_1 + \mathbf{D}\boldsymbol{\varepsilon}_1 \bullet \mathbf{L}_{11}(\dot{\mathbf{u}}, \mathbf{u}_1) + \boldsymbol{\sigma}_1 \bullet \mathbf{L}_{11}(\dot{\mathbf{u}}, \mathbf{u}_1) + \dot{\boldsymbol{\sigma}} \bullet \mathbf{L}_2(\mathbf{u}_1) = 0. \tag{130}$$

Equation (130) is the additional scalar equation needed to supplement (95) at $\lambda = \lambda_c$.

Bifurcation point. For a bifurcation point, from (118) we have

$$\mathbf{f} \bullet \mathbf{u}_1 = 0, \tag{131}$$

so that, from (122) and (123) we get

$$\mathbf{D}_s(\boldsymbol{\varepsilon}^c - \boldsymbol{\varepsilon}^i) \bullet \boldsymbol{\varepsilon}_1 = 0, \tag{132}$$

and λ_{cs} in (124) is not defined. To find λ_{cs} for the bifurcation case we consider an energy functional obtained by setting $\delta\mathbf{u} = \mathbf{u}_1$, $\delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1$ in (115c)

$$E = \boldsymbol{\sigma}_1 \bullet \boldsymbol{\varepsilon}_1 + \boldsymbol{\sigma} \bullet \mathbf{L}_2(\mathbf{u}_1) = 0. \tag{133}$$

Differentiating E with respect to the critical path parameter we have

$$\dot{E} = \dot{\boldsymbol{\sigma}}_1 \bullet \boldsymbol{\varepsilon}_1 + \boldsymbol{\sigma}_1 \bullet \dot{\boldsymbol{\varepsilon}}_1 + \dot{\boldsymbol{\sigma}} \bullet \mathbf{L}_2(\mathbf{u}_1) + 2\boldsymbol{\sigma}^c \bullet \mathbf{L}_{11}(\mathbf{u}_1, \dot{\mathbf{u}}_1) = 0. \tag{134}$$

We also have, by differentiating (115a) and (115b)

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}_1 &= \mathbf{L}_1(\dot{\mathbf{u}}_1) + \mathbf{L}_{11}(\dot{\mathbf{u}}, \mathbf{u}_1) + \mathbf{L}_{11}(\mathbf{u}^c, \dot{\mathbf{u}}_i), \\ \dot{\boldsymbol{\sigma}}_1 &= \mathbf{D}_s\boldsymbol{\sigma}_1 + \mathbf{D}\dot{\boldsymbol{\varepsilon}}_1. \end{aligned} \tag{135}$$

Using (135) we transform (134) into

$$\begin{aligned} \dot{E} = 2\boldsymbol{\sigma}_1 \bullet [\mathbf{L}_1(\dot{\mathbf{u}}_1) + \mathbf{L}_{11}(\mathbf{u}, \dot{\mathbf{u}}_1)] + 2\boldsymbol{\sigma}_1 \bullet \mathbf{L}_{11}(\dot{\mathbf{u}}, \mathbf{u}_1) + \mathbf{D}_s\boldsymbol{\varepsilon}_1 \bullet \boldsymbol{\varepsilon}_1 + \dot{\boldsymbol{\sigma}} \bullet \mathbf{L}_2(\mathbf{u}_1) \\ + 2\boldsymbol{\sigma}^c \bullet \mathbf{L}_{11}(\mathbf{u}_1, \dot{\mathbf{u}}_1) = 0. \end{aligned} \tag{136}$$

Next we use (115c) with $\delta\mathbf{u} = \dot{\mathbf{u}}_1$, $\delta\boldsymbol{\varepsilon} = \mathbf{L}_1(\dot{\mathbf{u}}_1) + \mathbf{L}_{11}(\mathbf{u}, \dot{\mathbf{u}}_1)$ to obtain

$$\boldsymbol{\sigma}_1 \bullet [\mathbf{L}_1(\dot{\mathbf{u}}_1) + \mathbf{L}_{11}(\mathbf{u}^c, \dot{\mathbf{u}}_1)] + \boldsymbol{\sigma}^c \bullet \mathbf{L}_{11}(\mathbf{u}_1, \dot{\mathbf{u}}_1) = 0, \tag{137}$$

so that we can simplify (136) to

$$\dot{E} = 2\boldsymbol{\sigma}_1 \bullet \mathbf{L}_{11}(\dot{\mathbf{u}}, \mathbf{u}_1) + \mathbf{D}_s\boldsymbol{\varepsilon}_1 \bullet \boldsymbol{\varepsilon}_1 + \dot{\boldsymbol{\sigma}} \bullet \mathbf{L}_2(\mathbf{u}_1) = 0. \tag{138}$$

Along the critical path both the load and stiffness vary simultaneously, so that

$$\begin{aligned} \dot{\mathbf{u}} &= \lambda_{cs}\mathbf{u}'^c + \mathbf{u}_s^c, \\ \dot{\boldsymbol{\sigma}} &= \lambda_{cs}\boldsymbol{\sigma}'^c + \boldsymbol{\sigma}_s^c. \end{aligned} \tag{139}$$

Substituting from (139) into (138) and solving for λ_{cs} we get

$$\lambda_{cs} = -\frac{\mathbf{D}_s\boldsymbol{\varepsilon}_1 \bullet \boldsymbol{\varepsilon}_1 + 2\boldsymbol{\sigma}_1 \bullet \mathbf{L}_{11}(\mathbf{u}_s^c, \mathbf{u}_1) + \boldsymbol{\sigma}_s^c \bullet \mathbf{L}_2(\mathbf{u}_1)}{2\boldsymbol{\sigma}_1 \bullet \mathbf{L}_{11}(\mathbf{u}'^c, \mathbf{u}_1) + \boldsymbol{\sigma}^{c\prime} \bullet \mathbf{L}_2(\mathbf{u}_1)}. \tag{140}$$

Equation (140) requires the calculation of the sensitivity of the prebuckling state with respect to the stiffness. To avoid this calculation we can use an adjoint method by appending the equations of the prebuckling path, (87), (91) and (92), to E, with adjoint fields acting as Lagrange multipliers. That is, the augmented function E^* is

$$E^* = E+\boldsymbol{\sigma}^a \bullet [\boldsymbol{\varepsilon}-\mathbf{L}_1(\mathbf{u})-\tfrac{1}{2}\mathbf{L}_2(\mathbf{u})]+\boldsymbol{\varepsilon}^a \bullet [\boldsymbol{\sigma}-\mathbf{D}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^i)]+\lambda\mathbf{f}\bullet\mathbf{u}^a - \boldsymbol{\sigma}\bullet[\mathbf{L}_1(\mathbf{u}^a)+\mathbf{L}_{11}(\mathbf{u},\mathbf{u}^a)] = 0. \quad (141)$$

Differentiating (141) and using (138) we obtain

$$\dot{E}^* = 2\boldsymbol{\sigma}_1 \bullet \mathbf{L}_{11}(\dot{\mathbf{u}},\mathbf{u}_1)+\mathbf{D}_{,s}\boldsymbol{\varepsilon}_1\bullet\boldsymbol{\varepsilon}_1+\dot{\boldsymbol{\sigma}}\bullet\mathbf{L}_2(\mathbf{u}_1) +\boldsymbol{\sigma}^a\bullet[\dot{\boldsymbol{\varepsilon}}-\mathbf{L}_1(\dot{\mathbf{u}})-\mathbf{L}_{11}(\mathbf{u}^c,\dot{\mathbf{u}})]+\boldsymbol{\varepsilon}^a\bullet[\dot{\boldsymbol{\sigma}}-\mathbf{D}_{,s}(\boldsymbol{\varepsilon}^c-\boldsymbol{\varepsilon}^i)-\mathbf{D}\dot{\boldsymbol{\varepsilon}}] +\dot{\lambda}\mathbf{f}\bullet\mathbf{u}^a-\dot{\boldsymbol{\sigma}}\bullet[\mathbf{L}_1(\mathbf{u}^a)+\mathbf{L}_{11}(\mathbf{u}^c,\mathbf{u}^a)]-\boldsymbol{\sigma}^c\bullet\mathbf{L}_{11}(\dot{\mathbf{u}},\mathbf{u}^a) = 0. \quad (142)$$

We now substitute (139) for $\dot{\mathbf{u}}^c$ and $\dot{\boldsymbol{\sigma}}^c$, and use (97) to eliminate some of the resulting terms

$$\dot{E}^* = 2\boldsymbol{\sigma}_1\bullet\mathbf{L}_{11}(\mathbf{u}_s^c,\mathbf{u}_1)+\mathbf{D}_{,s}\boldsymbol{\varepsilon}_1\bullet\boldsymbol{\varepsilon}_1+\boldsymbol{\sigma}_s^c\bullet\mathbf{L}_2(\mathbf{u}_1)+\lambda_{cs}[2\boldsymbol{\sigma}_1\bullet\mathbf{L}_{11}(\mathbf{u}^{c\prime},\mathbf{u}_1)+\boldsymbol{\sigma}^{c\prime}\bullet\mathbf{L}_2(\mathbf{u}_1)] +\boldsymbol{\sigma}^a\bullet[\boldsymbol{\varepsilon}_s^c-\mathbf{L}_1(\mathbf{u}_s^c)-\mathbf{L}_{11}(\mathbf{u}^c,\mathbf{u}_s^c)]+\boldsymbol{\varepsilon}^a\bullet[\boldsymbol{\sigma}_s^c-\mathbf{D}_{,s}(\boldsymbol{\varepsilon}^c-\boldsymbol{\varepsilon}^i)-\mathbf{D}\boldsymbol{\varepsilon}_s^c] -\boldsymbol{\sigma}_s^c\bullet[\mathbf{L}_1(\mathbf{u}^a)+\mathbf{L}_{11}(\mathbf{u}^c,\mathbf{u}^a)]-\boldsymbol{\sigma}^c\bullet\mathbf{L}_{11}(\mathbf{u}_s^c,\mathbf{u}^a) = 0. \quad (143)$$

To eliminate all derivatives of prebuckling response with respect to s we require the adjoint state to satisfy

$$\begin{aligned}
&\boldsymbol{\varepsilon}^a = \mathbf{L}_1(\mathbf{u}^a)+\mathbf{L}_{11}(\mathbf{u}^c,\mathbf{u}^a)-\mathbf{L}_2(\mathbf{u}_1),\\
&\boldsymbol{\sigma}^a = \mathbf{D}\boldsymbol{\varepsilon}^a,\\
&\boldsymbol{\sigma}^a\bullet\delta\boldsymbol{\varepsilon}-2\boldsymbol{\sigma}_1\bullet\mathbf{L}_{11}(\mathbf{u}_1,\delta\mathbf{u})+\boldsymbol{\sigma}^c\bullet\mathbf{L}_{11}(\mathbf{u}^a,\delta\mathbf{u}) = 0.
\end{aligned} \quad (144)$$

Then (143) becomes

$$\dot{E}^* = \mathbf{D}_{,s}\boldsymbol{\varepsilon}_1\bullet\boldsymbol{\varepsilon}_1+\lambda_{cs}[2\boldsymbol{\sigma}_1\bullet\mathbf{L}_{11}(\mathbf{u}^{c\prime},\mathbf{u}_1)+\boldsymbol{\sigma}^{c\prime}\bullet\mathbf{L}_2(\mathbf{u}_1)]-\mathbf{D}_{,s}(\boldsymbol{\varepsilon}^c-\boldsymbol{\varepsilon}^i)\bullet\boldsymbol{\varepsilon}^a = 0, \quad (145)$$

so that

$$\lambda_{cs} = \frac{\mathbf{D}_{,s}(\boldsymbol{\varepsilon}^c-\boldsymbol{\varepsilon}^i)\bullet\boldsymbol{\varepsilon}^a-\mathbf{D}_{,s}\boldsymbol{\varepsilon}_1\bullet\boldsymbol{\varepsilon}_1}{2\boldsymbol{\sigma}_1\bullet\mathbf{L}_{11}(\mathbf{u}'^c,\mathbf{u}_1)+\boldsymbol{\sigma}'^c\bullet\mathbf{L}_2(\mathbf{u}_1)}. \quad (146)$$

The sensitivity formulae (124), (127) and (146) are equivalent to the respective formulae (28), (29) and (68) derived in the previous section by starting from the potential energy function. Examples of the application of these formulae are presented in Haftka *et al.* (1990).

5. SENSITIVITY OF NON-POTENTIAL DISCRETE SYSTEMS

In this section, we shall discuss the case of discrete structures for which the tangent stiffness matrix is not symmetric. This case usually occurs for non-conservative loads such as friction or fluid pressure acting on structural elements.

The equations of equilibrium are not generated by a potential energy functional and may be written as

$$\mathbf{f}(\mathbf{u},s) = \lambda\mathbf{p}(s), \quad (147)$$

where λ denotes the proportional loading amplitude, s is a design or imperfection parameter,

$\mathbf{u}$ is the generalized displacement, and $\mathbf{f}$ and $\mathbf{p}$ denote the internal and external loads, respectively. Differentiating (147) with respect to a general path parameter η we get

$$J\dot{\mathbf{u}}+\mathbf{f}_{,s}\dot{s} = \dot{\lambda}\mathbf{p}+\lambda\mathbf{p}_{,s}\dot{s}, \tag{148}$$

where J is the Jacobian $\mathbf{f}_u$ (also known as the tangential stiffness matrix). Differentiating once more we obtain

$$J\ddot{\mathbf{u}}+\breve{J}(\mathbf{u},\dot{\mathbf{u}})\dot{\mathbf{u}}+2\dot{s}J_{,s}\dot{\mathbf{u}}+\mathbf{f}_{,s}\ddot{s}+\mathbf{f}_{,ss}\dot{s}^2 = \ddot{\lambda}\mathbf{p}+2\dot{\lambda}\dot{s}\mathbf{p}_{,s}+\lambda\mathbf{p}_{,ss}\dot{s}^2+\lambda\mathbf{p}_{,s}\ddot{s}, \tag{149}$$

where $\breve{J}$ is the matrix

$$\breve{J}(\mathbf{u},\dot{\mathbf{u}}) = \sum_i \frac{\partial J}{\partial u_i}\dot{u}_i. \tag{150}$$

We can specialize these equations for the case when the path parameter is a loading parameter and when the path parameter is the stiffness parameter s. For the former, (148) and (149) become

$$J\mathbf{u}' = \lambda'\mathbf{p}, \tag{151}$$

$$J\mathbf{u}''+J'\mathbf{u}' = \lambda''\mathbf{p}, \tag{152}$$

with

$$J' = \breve{J}(\mathbf{u},\mathbf{u}'). \tag{153}$$

For the case when s is the path parameter we get

$$J\mathbf{u}_{,s}+\mathbf{f}_{,s} = \lambda\mathbf{p}_{,s}, \tag{154}$$

$$J\mathbf{u}_{,ss}+J^s\mathbf{u}_{,s}+2J_{,s}\mathbf{u}_{,s}+\mathbf{f}_{,ss} = \lambda\mathbf{p}_{,ss}, \tag{155}$$

$$\mathbf{J}^s = \breve{\mathbf{J}}(\mathbf{u},\mathbf{u}_{,s}). \tag{156}$$

At a critical point J becomes singular, and (151) cannot be solved for the load-incremental state. The buckling mode $\mathbf{u}_1$ is the right eigenvector of J

$$J\mathbf{u}_1 = 0. \tag{157}$$

When the tangential stiffness matrix J is not symmetric we need also the left eigenvector $\mathbf{v}_1$ of J which is, in general, different from $\mathbf{u}_1$, and which satisfies

$$\mathbf{v}_1^T J = 0. \tag{158}$$

Premultiplying (151) by $\mathbf{v}_1^T$ we obtain

$$\lambda'\mathbf{v}_1^T\mathbf{p} = 0. \tag{159}$$

Equation (159) is satisfied at a limit point by $\lambda' = 0$, and at a bifurcation point by $\mathbf{v}_1^T\mathbf{p} = 0$.

At a limit point $\lambda' = 0$, and so by comparing (151) and (157) we note that $\mathbf{u}'$ is a scalar multiple of the eigenvector $\mathbf{u}_1$. Its magnitude can be determined only by specifying the path parameter. For example, if the path parameter is the jth coordinate, then we have the extra equation $u_j' = 1$ to permit us to calculate $\mathbf{u}'$. Premultiplying (152) by $\mathbf{v}_1^T$ we get

$$\lambda'' = \frac{\mathbf{v}_1^T J' \mathbf{u}'}{\mathbf{v}_1^T \mathbf{p}}. \tag{160}$$

Analytical computation of J' is tedious, and may be difficult to implement. Instead it is suggested that a semi-analytical method may be useful.

To implement the semi-analytical approach consider a product of the form $J'\mathbf{z}$ for a constant vector $\mathbf{z}$ where

$$J'\mathbf{z} = \frac{\mathrm{d}}{\mathrm{d}\eta}(J\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\partial f}{\partial u}\mathbf{z}\right) = \frac{\mathrm{d}}{\mathrm{d}\eta}\frac{\mathrm{d}}{\mathrm{d}e}[f(\mathbf{u}+e\mathbf{z})]_{e=0}. \tag{161}$$

If the structural analysis package that we use calculates the tangential stiffness matrix, J, we can make use of the first equality of (161) and have

$$J'\mathbf{u}_1 \cong \frac{J(\eta+\Delta\eta)-J(\eta)}{\Delta\eta}\mathbf{u}_1, \tag{162}$$

where η is the loading parameter. If the tangential stiffness matrix is not readily available, we can make use of the third equality

$$J'\mathbf{u}_1 \cong \frac{\mathrm{d}}{\mathrm{d}\eta}\left[\frac{f(\mathbf{u}+e\mathbf{u}_1)-f(\mathbf{u})}{e}\right], \tag{163}$$

where e is a small step-size parameter. Equation (162) requires that we find the tangential stiffness matrix at two close values of η. Equation (163) requires only the calculation of the internal loads in the structure for a given displacement field. The bracketed term in (163) is calculated by incrementing $\mathbf{u}$ by a small multiple of the buckling mode $\mathbf{u}_1$ and recalculating the internal load $\mathbf{f}$. The derivative with respect to η is obtained by finite differences by repeating the calculation at $\eta+\Delta\eta$.

Design or imperfection sensitivity can be calculated from (148) by considering a critical state path that follows limit points. Along the path we have $s = \eta$, $\dot{\lambda} = \lambda_{cs}$, so that (148) becomes

$$J\dot{\mathbf{u}}+\mathbf{f}_s = \lambda_{cs}\mathbf{p}+\lambda_c\mathbf{p}_s. \tag{164}$$

Premultiplying by the left eigenvector $\mathbf{v}_1^T$ we obtain

$$\lambda_{cs} = \frac{\mathbf{v}_1^T(\mathbf{f}_s-\lambda_c\mathbf{p}_s)}{\mathbf{v}_1^T\mathbf{p}}. \tag{165}$$

This derivative may also be conveniently implemented by a semi-analytical approach as

$$\lambda_{cs} = \frac{\mathbf{v}_1^T[\mathbf{r}(s+\Delta s)-\mathbf{r}(s)]}{\Delta s\mathbf{v}_1^T\mathbf{p}}, \tag{166}$$

where

$$\mathbf{r} = \mathbf{f}-\lambda\mathbf{p} \tag{167}$$

is the force residual. Note that $\mathbf{r}(s+\Delta s)$ is calculated without changing the load or displacement field obtained at the nominal point s. Equation (149) is specialized for the critical path as

$$J\ddot{\mathbf{u}}+(\breve{J}+2J_s)\dot{\mathbf{u}}+\mathbf{f}_{ss}=\lambda_{css}\mathbf{p}+2\lambda_{cs}\mathbf{p}_s+\lambda\mathbf{p}_{ss}. \tag{168}$$

Premultiplying by $\mathbf{v}_1^T$ we arrive at the formula

$$\lambda_{css}=\frac{\mathbf{v}_1^T(\breve{J}+2J_s)\dot{\mathbf{u}}+\mathbf{v}_1^T\mathbf{f}_{ss}-2\lambda_{cs}\mathbf{v}_1^T\mathbf{p}_s-\lambda_c\mathbf{v}_1^T\mathbf{p}_{ss}}{\mathbf{v}_1^T\mathbf{p}}. \tag{169}$$

To evaluate (169) we need to solve (164) for $\dot{\mathbf{u}}$ at $\lambda=\lambda_c$, where J is singular. We need one additional condition to make the solution unique. This condition is obtained by differentiating (157) along the critical path to obtain

$$\breve{J}(\mathbf{u},\dot{\mathbf{u}})\mathbf{u}_1+J_s\mathbf{u}_1+J\dot{\mathbf{u}}_1=0. \tag{170}$$

Premultiplying (170) by $\mathbf{v}_1^T$ we get

$$\mathbf{v}_1^T\breve{J}(\mathbf{u},\dot{\mathbf{u}})+\mathbf{v}_1^TJ_s\mathbf{u}_1=0, \tag{171}$$

which is the additional equation required to calculate $\dot{\mathbf{u}}$.

For a bifurcation point $\mathbf{v}_1^T\mathbf{p}=0$, and by premultiplying (154) by $\mathbf{v}_1^T$ we see that both the numerator and denominator of (165) are zero, and λ_{cs} cannot be calculated from (165). Note that it would appear that we can get the numerator of (165) to be zero even for a limit point in exactly the same way. However, for a limit point there is no solution for $\mathbf{u}_s$, so that the right hand side of (154) is inconsistent with the singular system.

To calculate the sensitivity of the bifurcation load we define an energy function

$$0=E=\mathbf{v}_1^TJ\mathbf{u}_1+\boldsymbol{\mu}^T(\mathbf{f}-\lambda\mathbf{p}), \tag{172}$$

where $\boldsymbol{\mu}$ is a Lagrange multiplier vector used to enforce prebuckling equilibrium. Differentiating (172) with respect to the critical path parameter we get

$$\mathbf{v}_1^T\breve{J}(\mathbf{u}_o^c,\dot{\mathbf{u}}_o)\mathbf{u}_1+\mathbf{v}_1^TJ_s\mathbf{u}_1+\boldsymbol{\mu}^T(J\dot{\mathbf{u}}_o+\mathbf{f}_s-\lambda_{cs}\mathbf{p}-\lambda_c\mathbf{p}_s)=0, \tag{173}$$

where $\mathbf{u}_o$ denotes the prebuckling state. Along the critical path both the load and the stiffness vary simultaneously so that

$$\dot{\mathbf{u}}_0=\lambda_{cs}\mathbf{u}'_0+\mathbf{u}_{0s}, \tag{174}$$

so that (173) may be written as

$$\lambda_{cs}\mathbf{v}_1^T\breve{J}(\mathbf{u}_0^c,\mathbf{u}'_0)\mathbf{u}_1+\mathbf{v}_1^T\breve{J}(\mathbf{u}_0^c,\mathbf{u}_{0s})\mathbf{u}_1+\mathbf{v}_1^TJ_s\mathbf{u}_1+\boldsymbol{\mu}^T[\lambda_{cs}J\mathbf{u}'_0+J\mathbf{u}_{0s}+\mathbf{f}_s-\lambda_{cs}\mathbf{p}-\lambda_c\mathbf{p}_s]=0. \tag{175}$$

We now have two options for calculating λ_{cs} from (175). We can set $\boldsymbol{\mu}$ to zero and get

$$\lambda_{cs}=-\frac{\mathbf{v}_1^T[\breve{J}(\mathbf{u}_0^c,\mathbf{u}_{0s})+J_s]\mathbf{u}_1}{\mathbf{v}_1^T\breve{J}(\mathbf{u}_0^c,\mathbf{u}'_0)\mathbf{u}_1}. \tag{176}$$

This form requires the calculation of the prebuckling sensitivity $\mathbf{u}_{0s}$. Instead we can define $\boldsymbol{\mu}$ to zero out the coefficient of $\mathbf{u}_{0s}$ in (175)

$$\mathbf{v}_1^TN(\mathbf{u}_0^c,\mathbf{u}_1)+\boldsymbol{\mu}^TJ=0, \tag{177}$$

where

$$N_{ij}(\mathbf{u}_0, u_1) = \frac{\partial J^c_{kj}}{\partial u_i} u_{1k}. \tag{178}$$

Then, from (175) we get

$$\lambda_{cs} = -\frac{\mathbf{v}_1^T J_s \mathbf{u}_1 + \boldsymbol{\mu}^T(\mathbf{f}_s - \lambda_c \mathbf{p}_s)}{\mathbf{v}_1^T J(\mathbf{u}^c_0, \mathbf{u}'_0)\mathbf{u}_1 + \boldsymbol{\mu}^T(J\mathbf{u}'_0 - \mathbf{p})}. \tag{179}$$

The term $\mathbf{v}_1^T N$ in (177) appears to require much algebra and computation for implementation. However, from its definition we have

$$(\mathbf{v}_1^T N)_j = \mathbf{v}_{1i} \frac{\partial J_{kj}}{\partial u_i} u_{1k}, \tag{180}$$

so that

$$\mathbf{v}_1^T N = \mathbf{u}_1^T \frac{\mathrm{d}}{\mathrm{d}e}[J(\mathbf{u}^c_0 + e\mathbf{v}_1)]|_{e=0} \cong \mathbf{u}_1^T \frac{J(\mathbf{u}^c_0 + e\mathbf{v}_1) - J(\mathbf{u}^c_0)}{e}. \tag{181}$$

When J is symmetric, then $\mathbf{u}_1 = \mathbf{v}_1$ and $\mathbf{u}_1^T J(\mathbf{u}^c_0) = 0$. Furthermore

$$J\dot{\mathbf{u}}_1 = \frac{\mathrm{d}}{\mathrm{d}e_1} \mathbf{f}(\mathbf{u} + e_1 \mathbf{u}_1)|_{e_1=0} \tag{182}$$

so

$$(\mathbf{v}_1^T N)^T = \frac{\partial^2}{\partial e\, \partial e_1} \mathbf{f}[\mathbf{u}^c_0 + e\mathbf{u}_1 + e_1 \mathbf{u}_1]. \tag{183}$$

Equation (181) shows that the term can be approximated with two evaluations of the Jacobian, while (183) shows that for the symmetric case no evaluation of the Jacobian is necessary.

6. EXAMPLE

6.1. *Example 1: sensitivity of a redundant beam structure*

The first example is a redundant beam system where the prebuckling state depends on the stiffness distribution. The axial force N and the bending moment M are the generalized stresses, with the axial strain ε and the curvature κ as the conjugate generalized strains. The strain displacement relations (87) now have the form

$$\varepsilon = v_x + \tfrac{1}{2} w_x^2 = e + \gamma, \quad \kappa = -w_{xx}, \tag{184}$$

where v denotes the axial displacement, w the lateral displacement, and the x-subscript denotes differentiation with respect to the axial coordinate along the member. Hooke's law is

$$N = EA(\varepsilon - \varepsilon^i), \quad M = EI\kappa, \tag{185}$$

where EA and EI are the extensional and flexural stiffnesses of the member, E, A and I are Young's modulus, cross-sectional area and moment of inertia, respectively, and ε^i denotes the initial strain.

The equation of equilibrium (92) now becomes

$$\sum_i \int_0^{l_i} (M\delta\kappa + N\delta\varepsilon)\,\mathrm{d}x_i = \lambda \mathbf{q} \bullet \delta \mathbf{u}, \tag{186}$$

where l_i is the length of the ith beam in the system. The incremental form (95c) is

$$\sum_i \int_0^{l_i} (\dot{M}\delta\kappa + \dot{N}\delta\varepsilon + N\dot{w}_x\delta w_x)\,\mathrm{d}x_i = \dot{\lambda} \mathbf{q} \cdot \delta \mathbf{u}. \tag{187}$$

For the prebuckling calculations it is typical to neglect the nonlinear term γ in the strain displacement relation (184). As a result, the prebuckling response is linear in the load amplitude λ

$$M = \lambda M' + M^i, \quad N = \lambda N' + N^i, \tag{188}$$

where M^i and N^i are the stress resultants due to initial strains, and where M' and N' satisfy a linear equation corresponding to (97c) with $\eta = \lambda$ and the $\mathbf{L}_{11}$ term neglected

$$\sum_i \int_0^{l_i} (M'\delta\kappa + N'\delta\varepsilon)\,\mathrm{d}x_i = \mathbf{q} \bullet \delta \mathbf{u}. \tag{189}$$

The buckling mode equations (115) become

$$\varepsilon_1 = u_{x1} + w w_1, \quad \kappa_1 = -w_{xx1}, \quad N_1 = EA\varepsilon_1, \quad M_1 = EI\kappa_1,$$
$$\sum_i \int_0^{l_i} (M_1\delta\kappa + N_1\delta\varepsilon + N w_{x1}\delta w_x)\,\mathrm{d}x_i = 0. \tag{190}$$

The second term in eqn (190c) is often neglected because it is zero when the prebuckling state is momentless.

For this example we assume that the buckling is of the bifurcation rather than limit-load type. This occurs when the prebuckling state is momentless or when the prebuckling state has some symmetry which is destroyed by buckling. In this case the sensitivity of the buckling load is given by (140) which becomes

$$\lambda_{cs} = -\frac{\sum_i \int_0^{l_i} [(EI)_{,s}\kappa_1^2 + (EA)_{,s}\varepsilon_{x1}^2 + 2N_1 w_{xs} w_x + N_s w_{x1}^2]\,\mathrm{d}x_i}{\sum_i \int_0^{l_i} (2N_1 w'_x w_{x1} + N' w_{x1}^2)\,\mathrm{d}x_i}. \tag{191}$$

If, as is commonly done, the prebuckling bending w and buckling mode axial response (N_1, ε_{x1}) are neglected, then eqn (191) takes the more familiar form of

$$\lambda_{cs} = -\frac{\sum_i \int_0^{l_i} [(EI)_{,s}\kappa_1^2 + N_s w_{x1}^2]\,\mathrm{d}x_i}{\sum_i \int_0^{l_i} N' w_{x1}^2\,\mathrm{d}x_i}. \tag{192}$$

Equations (191) and (192) require the calculation of the derivatives of the prebuckling response (N_s and w_{xs}). The adjoint method permits us to avoid this calculation. The adjoint fields satisfy (144) which become

$$\varepsilon^a = v_x^a + w_x^c w_x^a - w_{x1}^2, \quad \kappa^a = -w_{xx}^a, \quad N^a = EA\varepsilon^a, \quad M^a = EI\kappa^a,$$

$$\sum_i \int_0^{l_i} (N^a \delta\varepsilon + M^a \delta\kappa - 2N_1 w_{x1} \delta w_x + N^c w_x^a \delta w_x)\, \mathrm{d}x_i = 0. \tag{193}$$

Then the buckling load derivative is given by (146) as

$$\lambda_{cs} = \frac{\sum_i \int_0^{l_i} [(EA)_{,s}(\varepsilon^c - \varepsilon^i)\varepsilon^a + (EI)_{,s}\kappa^c\kappa^a - (EA)_{,s}\varepsilon_1^2 - (EI)_{,s}\kappa_1^2]\, \mathrm{d}x_i}{\sum_i \int_0^{l_i} (2N_1 w'_x w_{x1} + N' w_{x1}^2)\, \mathrm{d}x_i}. \tag{194}$$

Neglecting prebuckling bending and buckling mode axial response, (193) and (194) become

$$\varepsilon^a = v_x^a - w_{x1}^2, \quad \kappa^a = -w_{xx}^a, \quad N^a = EA\varepsilon^a, \quad M^a = EI\kappa^a,$$

$$\sum_i \int_0^{l_i} (N^a \delta\varepsilon + M^a \delta\kappa + N^c w_x^a \delta w_x)\, \mathrm{d}x_i = 0, \tag{195}$$

and

$$\lambda_{cs} = \frac{\sum_i \int_0^{l_i} [(EA)_{,s}(\varepsilon^c - \varepsilon^i)\varepsilon^a + (EI)_{,s}\kappa^c\kappa^a - (EA)_{,s}\kappa_1^2]\, \mathrm{d}x_i}{\sum_i \int_0^{l_i} N' w_{x1}^2\, \mathrm{d}x_i}. \tag{196}$$

As an example, consider a simple two-bar structure (Fig. 3), loaded through a rigid plate moving vertically. The structure is subjected to load p and temperature differential ΔT applied to bar A. The initial strain $\varepsilon^i = \alpha\Delta T$ in bar A generates the following self-equilibrated force system

$$N_A^i = -\alpha E\Delta T \frac{A_A A_B}{A_A + A_B}, \quad N_B^i = \alpha E\Delta T \frac{A_A A_B}{A_A + A_B}, \tag{197}$$

where α is the coefficient of thermal expansion. The external load p generates member forces

$$N_A = \lambda N'_A = -p\frac{A_A}{A_A + A_B}, \quad N_B = \lambda N'_B = -p\frac{A_B}{A_A + A_B}, \quad \lambda = p. \tag{198}$$

As the bars are hinged at both ends, the buckling mode takes the form $u_1 = 0$, $w_1 = \sin(\pi x/l)$, and the critical loads of bar A and bar B are

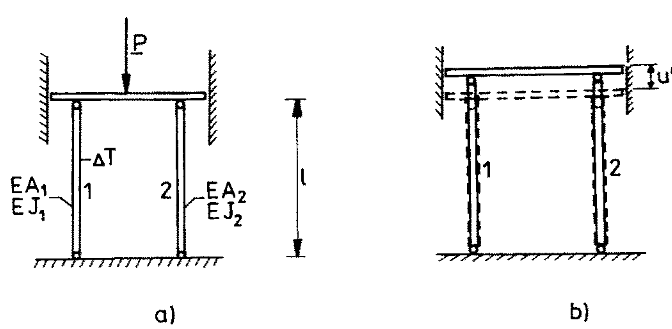

Fig. 3. (a) Two bar structure. (b) Adjoint structure.

$$\lambda_{cA} = \frac{\pi^2 EI_A}{l^2}\frac{A_A + A_B}{A_A} - \alpha E\Delta T A_B,$$

$$\lambda_{cB} = \frac{\pi^2 EI_B}{l^2}\frac{A_A + A_B}{A_B} + \alpha E\Delta T A_A. \tag{199}$$

Assume that $\Delta T > 0$ so that bar A buckles first. We also assume the moment of inertia to be related to the area as

$$I = \beta A^n, \tag{200}$$

where β is a constant factor and n depends on the cross-sectional form and its variation. Then

$$\frac{\partial \lambda_c}{\partial A_A} = \frac{\pi^2 EI_A}{l^2}\left[\frac{n(A_A + A_B)}{A_A^2} - \frac{A_B}{A_A^2}\right],$$

$$\frac{\partial \lambda_c}{\partial A_B} = \frac{\pi^2 EI_A}{l^2 A_A} - \alpha E\Delta T. \tag{201}$$

Equations (201), obtained directly, can now be used to check (192) which becomes

$$\frac{\partial \lambda_c}{\partial A_A} = -\frac{\int_0^l \left(E\frac{\partial I_A}{\partial A_A}\kappa_1^2 + \frac{\partial N_A}{\partial A_A} w_{x1}^2\right)\mathrm{d}x_A}{\int_0^l N'_A w_{x1}^2\,\mathrm{d}x_A},$$

$$\frac{\partial \lambda_c}{\partial A_B} = -\frac{\int_0^l \frac{\partial N_A}{\partial A_B} w_{x1}^2\,\mathrm{d}x_A}{\int_0^l N'_A w_{x1}^2\,\mathrm{d}x_A}. \tag{202}$$

The derivatives $\partial N_A/\partial A_A$ and $\partial N_A/\partial A_B$ in (202) need to be evaluated at $\lambda = \lambda_c$, and from (197) and (198)

$$\frac{\partial N_A}{\partial A_A} = -\alpha E\Delta T\left(\frac{A_B}{A_A + A_B}\right)^2 - \frac{\lambda_c A_B}{(A_A + A_B)^2} = -\frac{\pi^2 EI_A A_B}{A_A(A_A + A_B)l^2},$$

$$\frac{\partial N_A}{\partial A_B} = -\alpha E\Delta T\left(\frac{A_A}{A_A + A_B}\right)^2 + \frac{\lambda_c A_A}{A_A + A_B} = -\alpha E\Delta T\left(\frac{A_A}{A_A + A_B}\right) + \frac{\pi^2 I_A}{(A_A + A_B)l^2}. \tag{203}$$

With member 1 buckling

$$w_1 = \sin\left(\frac{\pi x}{l}\right), \quad w_{x1} = \frac{\pi}{l}\cos\left(\frac{\pi x}{l}\right), \quad \kappa_1 = \frac{\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right), \tag{204}$$

and from (198)

$$\int_0^l N'_A w_{x1}^2\,\mathrm{d}x_A = -\frac{\pi^2}{2l}\frac{A_A}{A_B + A_A}. \tag{205}$$

Also

$$\int_0^l E\frac{\partial I_A}{\partial A_A}\kappa_1^2\,\mathrm{d}x_A = \frac{\pi^4 EnI_A}{2A_A l^3},$$

$$\int_0^l \frac{\partial N_A}{\partial A_A}w_{x1}^2\,\mathrm{d}x_A = \frac{\pi^4 EI_A}{2A_A(A_A+A_B)l^3},$$

$$\int_0^l \frac{\partial N_A}{\partial A_B}w_{x1}^2\,\mathrm{d}x_A = \frac{\pi^4 EI_A - \pi^2 E\alpha\Delta T A_A l^2}{2(A_A+A_B)l^3}. \tag{206}$$

Substituting from (206) into (202) we verify (201).

The adjoint method (196) becomes for our example

$$\frac{\partial\lambda_c}{\partial A_A} = \frac{\int_0^l \left[E(\varepsilon_A^c - \alpha\Delta T)\varepsilon_A^a - E\frac{\partial I_A}{\partial A_A}\kappa_1^2\right]\mathrm{d}x_A}{\int_0^l N'_A w_{x1}^2\,\mathrm{d}x_A}$$

$$\frac{\partial\lambda_c}{\partial A_B} = \frac{\int_0^l E\varepsilon_B^c\varepsilon_B^a\,\mathrm{d}x_B}{\int_0^l N'_A w_{x1}^2\,\mathrm{d}x_A}. \tag{207}$$

From (195) the adjoint field satisfies the strain displacement relation

$$\varepsilon_A^a = u_{xA}^a - w_{xA}^2 = u_{xA}^a - \frac{\pi^2}{l^2}\cos^2\frac{\pi x}{l}. \tag{208}$$

The second term on the right hand side of (208) is a variable initial strain term, which has the same influence on member forces as a constant initial strain of the magnitude

$$\varepsilon^i = \frac{1}{l}\int_0^l \frac{\pi^2}{l^2}\cos^2\frac{\pi x}{l}\,\mathrm{d}x = \frac{\pi^2}{2l^2}. \tag{209}$$

The resulting axial member forces are

$$N_A^a = -E\varepsilon^i\frac{A_A A_B}{A_A+A_B} = -\frac{\pi^2 E}{2l^2}\frac{A_A A_B}{A_A+A_B}, \quad N_B^a = -N_A^a. \tag{210}$$

The corresponding strains are

$$\varepsilon_A^a = \frac{N_A^a}{EA_A} = -\frac{\pi^2}{2l^2}\frac{A_B}{A_A+A_B}, \quad \varepsilon_B^a = \frac{N_B^a}{EA_B} = \frac{\pi^2}{2l^2}\frac{A_A}{A_A+A_B}. \tag{211}$$

To evaluate (207) we also need the strains in members A and B at the buckling load:

$$\varepsilon_A^c = \frac{N_A^c}{EA_A} + \alpha\Delta T = -\alpha\Delta T\frac{A_B}{A_A+A_B} - \frac{\lambda_c}{E}\frac{1}{A_A+A_B} + \alpha\Delta T$$

$$= \alpha\Delta T - \frac{\pi^2 I_A}{l^2 a_A}, \tag{212}$$

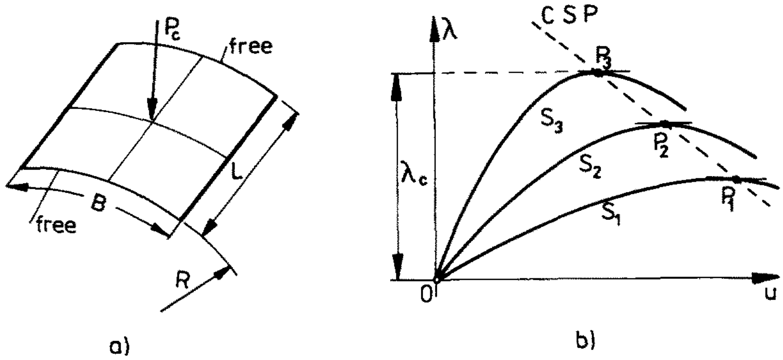

Fig. 4. Simply supported cylindrical shell with a central point load. ($R = 2540$ mm, $L = B = 504$ mm, $E = 3105$ N/mm^2, $v = 0.3$, $t = 6.35$ mm).

$$\varepsilon_B^c = \frac{N_B^c}{EA_B} = \alpha\Delta T \frac{A_A}{A_A+A_B} - \frac{\lambda_c}{E}\frac{1}{A_A+A_B}$$

$$= \alpha\Delta T - \frac{\pi^2 I_A}{l^2 A_A}. \tag{213}$$

Naturally, $\varepsilon_A = \varepsilon_B$. Substituting into (207) we get

$$\int_0^l E(\varepsilon_A^c - \alpha\Delta T)\varepsilon_A^a \, dx_A = -E\frac{\pi^2 I_A}{l^2 A_A}\left(-\frac{\pi^2}{2l^2}\frac{A_B}{A_A+A_B}\right)l = \frac{\pi^4 EI_A}{2A_A(A_B+A_A)l^3}, \tag{214}$$

which is the same as $\int_0^l (\partial N_A/\partial A_A)w_{x1}^2 \, dx_A$ in (206). Since the other terms in $(\partial\lambda_c/\partial A_A)$ in (202) are the same as in (207) we have confirmed that the adjoint and direct method give the same result for $\partial\lambda_c/\partial A_A$. Similarly for $\partial\lambda_c/\partial A_B$ in (207) we have

$$\int_0^l E\varepsilon_B^c\varepsilon_B^a \, dx_B = \left(E\alpha\Delta T - \frac{\pi^2 EI_A}{l^2 A_A}\right)\frac{\pi^2}{2l^2}\frac{A_A}{A_A+A_B}l = -\frac{\pi^4 EI_A - \pi^2 E\alpha\Delta T A_A l^2}{2(A_A+A_B)l^3} \tag{215}$$

which agrees with the third integral of (206).

6.2. *Example 2: shallow cylindrical shell subjected to point load*

The second example is a shallow cylindrical shell simply supported along two straight edges and loaded by a central load p (see Fig. 4). The shell parameters are $R = 2540$ mm, $L = B = 504$ mm, $E = 3104$ N/mm^2, $\gamma = 0.3$ and the thickness $t = 6.53$ mm.

This example, taken from Haftka (1993) is intended to demonstrate the implementation of (166) with a general purpose finite element program.

Equations (165) and (166) for the derivative of the limit-load have to be implemented right at the limit-load itself. However, it can be difficult to obtain structural response exactly at the limit-load, so that the calculation must be implemented away from the limit-load. Therefore, the derivative based on (166) was calculated for several points near the limit-load along the prebuckling path. Derivatives with respect to the thickness and Poisson's ratio are presented in Table 1. These are given as logarithmic derivatives

$$\frac{\partial^l \lambda_c}{\partial s} = \frac{\partial(\ln \lambda_c)}{\partial(\ln s)} = \frac{s}{\lambda_c}\frac{\partial \lambda_c}{\partial s} = \frac{s}{\lambda_c}\lambda_{cs}. \tag{216}$$

One advantage of logarithmic derivatives is they show the underlying functional relationship. Thus, if λ_c is proportional to s^n then $\partial^l \lambda c/\partial s = n$. Table 1 shows that the buckling load is approximately proportional to t^2. The value of about 0.1 for the logarithmic derivative with respect to v is consistent with a proportionality to $(1-v^2)^{-1/2}$. The log-

Table 1. Logarithmic derivatives of limit-load amplitude with respect to shell thickness and Poisson's ratio for cylindrical shell

$\frac{\lambda}{\lambda_c}$	$\frac{t}{\lambda_c}\frac{\partial \lambda_c}{\partial t}$		$\frac{v}{\lambda_c}\frac{\partial \lambda_c}{\partial v}$	
	FD	SA	FD	SA
0.9996	2.067	2.073	0.1095	0.1099
0.995		2.103		0.1136
0.973		2.180		0.1216
0.928		2.294		0.1325
0.885		2.435		0.1454

arithmic derivative of this term is $v^2/(1-v^2) = 0.099$ for $v = 0.3$. In Table 1 the semi-analytical (SA) derivative form (166) is compared to a forward-difference (FD) derivative. The results indicate that substantial error can occur if the derivative is calculated too far away from the limit load.

7. CONCLUDING REMARKS

The present paper provides a variational approach to first and second order sensitivity analysis of non-linear structures undergoing deformation in regular and critical states. The direct and adjoint approaches were discussed. It was shown that there is a close connection between post-critical deformation analysis and sensitivity. In fact, for the case of symmetric buckling, the adjoint fields are identical to second order post-critical fields following from the asymptotic expansion in terms of deformation parameter. It is believed that the present approach will prove useful in analyzing structural redesign or optimization, and also in assessing the effect of damage or structural imperfections on the critical load value.

One important issue is the sensitivity of post-critical response to variation of structural parameters. This problem will be discussed in a separate paper.

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